Moments and Center of Mass

Archimedes used "moving power" to describe the effect of a lever in moving a mass on the other end. He is famous for the quote "Give me a lever long enough and a fulcrum on which to place it, and I shall move the world." We will define this idea as "moment"

We will look at moments about
  A point—1 dimension
  A line—2 dimensions
  A plane—3 dimensions

**Moment = (mass) * (distance)**

I. One-Dimensional System

The measure of the tendency of this system to rotate about the origin is the **moment about the origin**. \( M_0 = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n \)

\( M_0 \) = moment about the origin. When \( M_0 = 0 \), the system is said to be in **equilibrium**.

**Center of Mass**: point where the system is in equilibrium, denoted by \( \bar{x} \)

\[ \bar{x} = \frac{M_0}{m} \text{, where } m = m_1 + m_2 + \cdots + m_n \text{ is the total mass of the system.} \]

Example 1: Find the **moment about the origin** and **center of mass** of a system of four objects with masses 10 g, 45 g, 32 g, and 24 g that are located at the points −4, 1, 3, and 8, respectively, on the x-axis.

Example 2: If a 20 pound child and a 60 pound child sit on the ends of a seesaw that is 5 feet long, where must the fulcrum be located so that the beam will balance?
II. Two-Dimensional System

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

1. The **moment about the y-axis** is $M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$.

2. The **moment about the x-axis** is $M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n$.

3. The **center of mass** $(\bar{x}, \bar{y})$ (or center of gravity) is

   $$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

   where $m = m_1 + m_2 + \cdots + m_n$

   is the total mass of the system.

**Example 3:** Find the center of mass of the given system of point masses.

<table>
<thead>
<tr>
<th>$m_i$</th>
<th>3</th>
<th>4</th>
<th>2</th>
<th>1</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x_i, y_i)$</td>
<td>$(-2, -3)$</td>
<td>$(5, 5)$</td>
<td>$(7, 1)$</td>
<td>$(0, 0)$</td>
<td>$(-3, 0)$</td>
</tr>
</tbody>
</table>

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III. Three-Dimensional System-Using geometry

Mass = (density)*(Area) or Mass = ($\rho$)*(Area).

A planar lamina (a thin plate) with uniform density $\rho$. If we take the density to equal 1, then the mass is numerically equal to the area.

**Example 4:** Find the center of mass of the region shown.
III. Three-Dimensional System-Irregular Laminas

$$\bar{x} = x \text{ and } \bar{y} = \frac{1}{2} (f(x) + g(x))$$

### Representative Thin Strip (rectangle):

1. **Area of representative rectangle:**
   $$\Delta A = (f(x) - g(x)) \Delta x \text{ (Height} \times \text{width)}$$

2. The **moment about the y-axis** is
   $$M_y = \bar{x} \Delta A = x(f(x) - g(x)) \Delta x$$

3. The **moment about the x-axis** is
   $$M_x = \bar{y} \Delta A = \frac{1}{2} (f(x) + g(x))(f(x) - g(x)) \Delta x$$
   $$= \frac{1}{2} ([f(x)]^2 - [g(x)]^2) \Delta x$$

4. The **center of mass** \((\bar{x}, \bar{y})\) (or center of gravity) is
   \(\bar{x} = x \text{ and } \bar{y} = \frac{1}{2} (f(x) + g(x))\)

### Full Region

1. **Area of region:**
   $$A = \int_a^b (f(x) - g(x)) \, dx$$

2. The **moment about the y-axis** is
   $$M_y = \int_a^b x(f(x) - g(x)) \, dx.$$  

3. The **moment about the x-axis** is
   $$M_x = \frac{1}{2} \int_a^b ([f(x)]^2 - [g(x)]^2) \, dx.$$  

4. The **center of mass** \((\bar{x}, \bar{y})\) (or center of gravity) is
   $$\bar{x} = \frac{M_y}{A} \text{ and } \bar{y} = \frac{M_x}{A}$$

Example 5: Find the center of mass of a lamina of uniform density \(\rho\) covering the region bounded by the parabola \(y = 4 - x^2\) and \(y = 0\). Let \(\rho = 1\) (Hint: symmetry)
Example 6: Find the center of mass of a lamina of uniform density $\rho$ covering the region bounded by the parabola $y = x^4$ and $y = x$. Let $\rho = 1$.
Example 7: Find the center of mass of a lamina of uniform density $\rho$ covering the region bounded by the parabola $x = y + 2$ and $x = y^2$. Let $\rho = 1$