Finding Distances

Example 1: \( y = 4 - x^2 \). Find the distance from the \( y \)-axis and \( x \)-axis in terms of \( x \) and \( y \).

\[
\begin{align*}
\text{Solve for } x: & \quad y = 4 - x^2 \\
x^2 & = 4 - y \\
x & = \pm \sqrt{4 - y} \\
\text{Use only + because of graph.}
\end{align*}
\]

\[
\begin{align*}
d_1 &= y = 4 - x^2 \\
d_2 &= x = \sqrt{4 - y} \\
\end{align*}
\]

Example 2: \( x = -y^2 + 4y \).

Find the distance from the \( y \)-axis in terms of \( x \) and \( y \).

\[
d = x = -y^2 + 4y
\]

Example 3:

Find the distance from \( y = 3 - \frac{x^2}{2} \) to the line: \( y = 1 \) in terms of \( x \) and \( y \).

\[
d = \text{Top - Bottom} = (3 - \frac{x^2}{2}) - 1 \text{ or } d = y - 1
\]

Example 4: Find the distance from \( y = 3 - \frac{x^2}{2} \) to the line: \( y = -1 \) in terms of \( x \) and \( y \).

\[
\begin{align*}
d &= y + 1 \\
d &= 3 - \frac{x^2}{2} + 1
\end{align*}
\]

Example 5: Find the distance from \( y = 2x \) to the line: \( x = 2 \) in terms of \( x \) and \( y \).

\[
\begin{align*}
d_1 &= 2 \\
\text{Solve for } x: & \quad \pm \sqrt{\frac{y}{2}} = x \\
d_2 &= x \\
\text{or } d &= 2 - x \\
\end{align*}
\]

\[
\begin{align*}
d &= 2 - \sqrt{\frac{y}{2}}
\end{align*}
\]
Example 6: Find the distance from \( y = 2x^2, x \geq 0 \) to the line: \( x = -3 \) terms of \( x \) and \( y \).

\[
d = 3 + \frac{y}{3}
\]

Example 7: Find the distance from \( y = 2x^2, x \geq 0 \) to the line: \( y = 8 \) terms of \( x \) and \( y \).

\[
d = 8 - y
\]

Example 7: Find the indicated distances:

\( y = -(x-2)^2 + 3 \)

\[
r = x \\
h = y = -(x-2)^2 + 3
\]

Example 8: Find the indicated distances:

\( x = -(y-2)^2, y = x, y = 6, y = -1 \)

\[
h = \text{right} - \text{left} \\
h = \frac{1}{2} - (-(y-2)^2) \\
h = y + (y+2)
\]

\[
r_1 = 6 - y \\
r_2 = y + 1
\]
Review from Geometry: The volume of a cylinder

\[ V = \pi r^2 h \]

Determine the Volume of a Solid of Revolution:
So, for the purposes of the derivation of the formula, let’s look at rotating the continuous function \( y = f(x) \) in the interval \([a, b]\) about the x-axis. Below is a sketch of a function and the solid of revolution we get by rotating the function about the x-axis.

Short animation: [https://youtu.be/i4L5XoUBD_Q](https://youtu.be/i4L5XoUBD_Q)

The volume of the disk (a cylinder) is given by: \( \Delta V = \pi r^2 \Delta x \). Approximating the volume of the solid by \( n \) disks of width \( \Delta x \) with radius \( r(x) \), produces:

\[
\text{Volume of solid} \approx \sum_{i=1}^{n} \pi [r(x_i)]^2 \Delta x
\]

This approximation appears to become better and better as \( \Delta x \to 0 \) \((n \to \infty)\). Therefore,

\[
\text{Volume of solid} = \lim_{\Delta x \to 0} \pi \sum_{i=1}^{n} [r(x_i)]^2 \Delta x = \pi \int_{a}^{b} [r(x)]^2 dx
\]
As seen in animation, we can rotate functions around the $y$-axis:

The radius is now a function of $y$.

Volume of solid $= \pi \int_{c}^{d} [r(y)]^2 dy$

Note: the radius is ALWAYS perpendicular to axis of rotation.

Example 1: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the $x$-axis. $y = 4 - x^2$ (p.453: 2)


$$V = \pi \int_{0}^{2} [r(x)]^2 dx = \pi \int_{0}^{2} (4 - x^2) dx$$

$$= \pi \left[ 16x - \frac{8x^3}{3} + \frac{x^4}{5} \right]_{0}^{2}$$

$$= \pi \left[ (16(2) - \frac{8(8)}{3} + \frac{32}{5}) - 0 \right] = \frac{256}{15} \pi$$

Example 2: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the $y$-axis. $y = 4 - x^2$

$$V = \pi \int_{0}^{4} [r(y)]^2 dy$$

$$V = \pi \int_{0}^{4} (4 - y^2) dy$$

$$V = \pi \left[ 4y - \frac{y^3}{3} \right]_{0}^{4}$$

$$V = \pi \left[ (16 - \frac{16}{3}) - 0 \right]$$

$$V = 8\pi$$

Solve for $x$

$y = 4 - x^2$

$x^2 = 4 - y$

$\chi = \sqrt{4 - y}$

only need $\chi = \sqrt{4 - y}$

right-hand side
Example 3: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the $y$-axis. \[ x = -y^2 + 4y \] (p.453: 10)

\[
V = \pi \int_{-2}^{4} (-(y^2 - 4y)^2) \, dy
\]

\[
V = \pi \int_{1}^{4} (y^4 - 8y^3 + 16y^2) \, dy
\]

\[
V = \pi \left[ \frac{y^5}{5} - \frac{8y^4}{4} + \frac{16y^3}{3} \right]_{1}^{4}
\]

\[
V = \pi \left[ \left( \frac{4^5}{5} - \frac{8(4)^4}{4} + \frac{16(4)^3}{3} \right) - \left( \frac{1}{5} - \frac{2}{4} + \frac{16}{3} \right) \right]
\]

\[
V = \pi \left[ \frac{512}{15} - \frac{53}{15} \right] = \frac{459}{15} \pi = \frac{153}{5} \pi
\]

Revolving about a line that is NOT the $x$ or $y$ axis.

Example 4: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the line $y = 1$: \[ y = 3 - \frac{x^2}{2} \]

\[
r = 3 - \frac{x^2}{2} - 1 = 2 - \frac{x^2}{2}
\]

\[
V = \pi \int_{-2}^{2} \left[ (2 - \frac{x^2}{2})^2 \right] \, dx = \pi \int_{-2}^{2} \left( 4 - 2x^2 + \frac{x^4}{4} \right) \, dx
\]

\[
V = \pi \left[ (4x - \frac{2x^3}{3} + \frac{x^5}{20}) \right]_{-2}^{2}
\]

\[
V = \pi \left[ \left( 8 - \frac{16}{3} + \frac{32}{20} \right) - \left( -8 + \frac{16}{3} - \frac{32}{20} \right) \right]
\]

\[
V = \pi \left[ \left( \frac{16}{5} \right) - \left( -\frac{16}{5} \right) \right] = \frac{128}{15} \pi
\]

Example 5: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the line $x = 2$: \[ y = 2x^2 \] (p.453: 12d)

\[
r = 2 - x
\]

\[
r = 2 - \sqrt{\frac{y}{2}}
\]

\[
r = 2 - \sqrt{\frac{y}{2}}
\]

\[
V = \pi \int_{0}^{8} (2 - 1) \, dy = \pi \int_{0}^{8} \left( 4 - 4\sqrt{\frac{y}{2}} \right) \, dy
\]

\[
V = \pi \left[ (4y - 4\sqrt{\frac{y}{2}} + \frac{y}{2}) \right]_{0}^{8}
\]

\[
V = \pi \left[ \left( 32 - \frac{8}{3\sqrt{2}} (8^{3/2} + 64) \right) - 0 \right] = \frac{16}{3} \pi
\]

Solve for $x$

\[
\frac{y}{2} = x^2
\]

\[
\frac{y}{2} = x
\]
The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative washer.

Volume of the washer = $\pi (R^2 - r^2)w$
Where $R = \text{the outer radius}$ and $r = \text{inner radius}$
If rotated around a horizontal axis, then

Volume of solid = $\pi \int_a^b ([R(x)]^2 - [r(x)]^2)dx$

If rotated around a vertical axis, then

Volume of solid = $\pi \int_c^d ([R(y)]^2 - [r(y)]^2)dy$

Example 1: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by the graphs: $y = x^2$, $y = \sqrt{x}$, about x-axis.

\[ V = \pi \int_0^1 \left( (\sqrt{x})^2 - (x^2)^2 \right) dx \]
\[ V = \pi \left[ \left( \frac{x^2}{2} - \frac{x^5}{5} \right) \right]_0^1 \]
\[ V = \pi \left[ \left( \frac{1}{2} - \frac{1}{5} \right) - 0 \right] = \frac{3}{10} \pi \]

(answer is $\frac{3\pi}{10}$)
Example 2: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by the graphs: $y = 2x^2$, $y = 0$, $x = 2$ about the line $y = 8$. (p.453: 12c)

Example 3: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by the graphs: $y = 2x^2$, $y = 0$, $x = 2$ about the y-axis. (p.453: 12a)

For the following problems, (a) Find the outer radius, $R$, and the inner radius, $r$, (b) find the limits of integration, and (c) set up the integral that gives the volume of the solid.

**Classwork**

1. $y = x^2$, $y = x$, about the y-axis.
2. $y = x^2$, $y = x$, about line $x = -1$.
3. $y = x^2$, $y = x$, about line $y = 3$.
4. $y = (x - 1)^2 + 1$, $y = 1$, $x = 0$, $x = 1$ about the x-axis.
5. $y = (x - 1)^2 + 1$, $y = 1$, $x = 0$, $x = 1$ about line $x = -1$.
6. $y = (x - 1)^2 + 1$, $y = 1$, $x = 0$, $x = 1$ about line $x = 2$.

\[ \text{Solve for } x: \quad y = 2x^2 \]