Review of Cal I

Trig Derivatives:
\[
\frac{d}{dx}[\sin u] = \cos u \quad \frac{d}{dx}[\cos u] = -\sin u \quad \frac{d}{dx}[\tan u] = \sec^2 u
\]
\[
\frac{d}{dx}[\sec u] = \tan u \sec u \quad \frac{d}{dx}[\cot u] = -\csc^2 u \quad \frac{d}{dx}[\csc u] = -\csc u \cot u
\]

Exponential and Natural Log Functions:
\[
\frac{d}{dx}[e^u] = e^u \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[\ln u] = \frac{1}{u} \frac{du}{dx} \quad \text{or} \quad \frac{u'(x)}{u}
\]

\[f(x) = e^{x^2+x}\]
\[f(x) = \ln(\sin(x))\]

Power Rule:
\[
\frac{d}{dx}[x^n] = nx^{n-1}
\]
\[f(x) = x^4 - 3x^3 + 6x^2 - \sqrt{x}\]

Product Rule:
\[
\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}
\]
\[f(x) = x^3 \cos x\]

Quotient Rule:
\[
\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad \text{OR} \quad \frac{d}{dx}\left[\frac{T}{B}\right] = \frac{BT' - TB'}{B^2}
\]
\[f(x) = \frac{x^3 - 1}{x^2 + 2}\]
Chain Rule: \[ \frac{d}{dx} [u]^n = nu^{n-1}u' \]

\[ f(x) = (3x^2 - 5x + 1)^4 \]

\[ f(x) = \sin^3 (5x) \]

\[ f(x) = \tan^5 \left( \sqrt{3x^2 - 1} \right) \]
Implicit Differentiation:
Find $\frac{dy}{dx}$, given $x^3 + 4x^2y^4 - 3y^5 = 9$

Find $\frac{dy}{dx}$, given $\cos(xy) + y^3 = 4$

Integrals: $\int f(x)\,dx = \text{a set of antiderivatives}$  
Why a set?

Review of Basic Forms

\[
\begin{align*}
\int k \, du &= ku + C \\
\int \sin(u) \, du &= -\cos(u) + C \\
\int \cos(u) \, du &= \sin(u) + C \\
\int \sec^2(u) \, du &= \tan(u) + C \\
\int \frac{1}{u} \, du &= \ln|u| + C \\
\int (x^3 - 3\sqrt{x} - \frac{1}{x^5}) \, dx &= \frac{x^4}{4} - \frac{3}{2}x^{3/2} + \frac{1}{4x^4} + C \\
\int x(5x^2 + 4)^3 \, dx &= \frac{1}{20}(5x^2 + 4)^4 + C
\end{align*}
\]
\[ \int \frac{6x^2}{(4x^2 - 9)^5} \, dx \quad \int \frac{\cos x}{\sin^3 x} \, dx \]

\[ \int \sqrt{\tan x \, \sec^2 x} \, dx \quad \int x \sin(6x^2) \, dx \]

\[ \int \frac{x}{x^2 + 1} \, dx \quad \int 2xe^{x^2} \, dx \]
Calculus I Review Worksheet

Find the derivative \( y' \) or \( \frac{dy}{dx} \) for each function.

1. \( y = (x^2 - 4x + 5)^{2/3} \)
2. \( y = \tan(2\pi x) \)
3. \( y = \cos(5x) \)
4. \( y = \sin^2(3x) \)
5. \( y = x^2 \tan x \)

Find the indefinite or definite integral:

1. \( \int x(4x^2 + 3)^4 \, dx \)
2. \( \int \left( x^5 - 2x^4 + \frac{3}{\sqrt{x}} - \frac{1}{x^3} \right) \, dx \)
3. \( \int \sec(5x) \tan(5x) \, dx \)
4. \( \int \frac{dx}{7x - 2} \)
5. \( \int \frac{\sin x}{\cos^3 x} \, dx \)
6. \( \int \sin 6x \, dx \)
7. \( \int \frac{dx}{\sqrt{1 - 3x}} \)
8. \( \int x \cos(x^2) \, dx \)
9. \( \int xe^{1-x^2} \, dx \)
10. \( \int \frac{x^3}{3\sqrt[3]{1 + x^4}} \, dx \)
Calculus I Review Worksheet-Answers

1. \[ y' = \frac{2(2x - 4)}{3(x^2 - 4x + 5)^{1/3}} \]
2. \[ y' = 2\pi \sec^2(2\pi x) \]
3. \[ y' = -5\sin(5x) \]
4. \[ y' = 6\sin(3x)\cos(3x) \]
5. \[ y' = x^2\sec^2(x) + 2x\tan(x) \]
6. \[ y' = \frac{-3\sin(3x)(x - \sin(3x)) - \cos(3x)(1 - 3\cos(3x))}{(x - \sin(3x))^2} \]

7. \[ \frac{dy}{dx} = \frac{4xy^2 - 3x^2}{15y^2 - 4x^2y} \]
8. \[ \frac{dy}{dx} = -\frac{2xy}{x^2 - 3\sec^2 y} \]
9. \[ y' = -3x^2\cos(x^3)\sin(\sin(x^3)) \]
10. \[ y' = 15(3x - 2)^4(2x + 3)^4 + 8(2x + 3)^3(3x - 2)^5 \]

| 1. \[ \frac{(4x^2 + 3)^5}{40} + C \] | 7. \[ -\frac{2}{3}\sqrt{1 - 3x} + C \] | 13. \[ \frac{1}{3}e^{3x} - e^x - e^{-x} + C \] |
| 2. \[ \frac{x^6 - 2x^5 + 3x^{4/3}}{5} + \frac{1}{4x^2} + C \] | 8. \[ \frac{1}{2}\sin(x^2) + C \] | 14. \[ \frac{\sin^7(x)}{7} + C \] |
| 3. \[ \frac{1}{5}\sec(5x) + C \] | 9. \[ -\frac{1}{2}e^{1-x^2} + C \] | 15. \[ \frac{1}{4}(\tan^2(2x)) + C \] |
| 4. \[ \frac{1}{7}\ln|7x - 2| + C \] | 10. \[ \frac{3}{8}(1 + x^4)^{2/3} + C \] | 16. \[ \frac{2}{3}(1 - 2x^2)^{3/2} + C \] |
| 5. \[ \frac{1}{2\cos^2 x} + C \] | 11. \[ -\ln|x + \cos x| + C \] | 17. \[ \frac{\sec^5(x)}{5} + C \] |
| 6. \[ -\frac{1}{6}\cos(6x) + C \] | 12. \[ \frac{1}{-3(x - 3)^3} + C \] | 18. \[ \ln|e^3 - 1| - \ln|e - 1| = \ln\left|\frac{e^3 - 1}{e - 1}\right| \] |
Area of Region Between Two Curves

Review from Cal 1: The area under a curve.

Write the integral or integrals to find the area of each region:

1.

2.

3.

Area Between 2 Curves:

4.
5. With respect to $x$

6. With respect to $y$

7.
Examples:
Sketch a graph of the region bounded by the given equations and find the area of the region:

1. \[ y = -x^2 + 3x + 1 \]
   \[ y = -x + 1 \]

2. \[ f(y) = \frac{y}{\sqrt{16-y^2}}, \quad g(y) = 0, \quad y = 3 \]
(Use Desmos.com for graph)

Finding Distances

Example 1: \( y = 4 - x^2 \). Find the distance from the \( y \)-axis and \( x \)-axis in terms of \( x \) and \( y \).

Example 2: \( x = -y^2 + 4y \).

Find the distance from the \( y \)-axis in terms of \( x \) and \( y \).

Example 3:

Find the distance from \( y = 3 - \frac{x^2}{2} \) to the line: \( y = 1 \) in terms of \( x \) and \( y \).

Example 4: Find the distance from \( y = 3 - \frac{x^2}{2} \) to the line: \( y = -1 \) in terms of \( x \) and \( y \).

Example 5: Find the distance from \( y = 2x^2 \) to the line: \( x = 2 \) in terms of \( x \) and \( y \).
Example 6: Find the distance from $y = 2x^2, x \geq 0$ to the line: $x = -3$ terms of $x$ and $y$.

Example 7: Find the distance from $y = 2x^2, x \geq 0$ to the line: $y = 8$ terms of $x$ and $y$.

Example 7: Find the indicated distances:

\[ y = -(x - 2)^2 + 3 \]

Example 8: Find the indicated distances:

\[ x = -(y - 2)^2, y = x, y = 6, y = -1 \]
Volume: The Disk Method

Review from Geometry: The volume of a cylinder

\[ V = \pi r^2 h \]

Determine the Volume of a Solid of Revolution:
So, for the purposes of the derivation of the formula, let’s look at rotating the continuous function \( y = f(x) \) in the interval \([a,b]\) about the \(x\)-axis. Below is a sketch of a function and the solid of revolution we get by rotating the function about the \(x\)-axis.

Short animation: https://youtu.be/i4L5XoUBD_Q

The volume of the disk (a cylinder) is given by: \( \Delta V = \pi r^2 \Delta x \). Approximating the volume of the solid by \(n\) disks of width \(\Delta x\) with radius \(r(x)\), produces:

\[
\text{Volume of solid} \approx \sum_{i=1}^{n} \pi [r(x_i)]^2 \Delta x
\]

This approximation appears to become better and better as \(\Delta x \to 0 \ (n \to \infty)\). Therefore,

\[
\text{Volume of solid} = \lim_{\Delta x \to 0} \pi \sum_{i=1}^{n} [r(x_i)]^2 \Delta x = \pi \int_{a}^{b} [r(x)]^2 \, dx
\]
As seen in animation, we can rotate functions around the $y$-axis:

The radius is now a function of $y$.

Volume of solid = $\pi \int_{c}^{d} [r(y)]^2 \, dy$

Note: the radius is ALWAYS perpendicular to axis of rotation.

Example 1: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the $x$-axis. $y = 4 - x^2$


Example 2: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the $y$-axis. $y = 4 - x^2$
Example 3: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the $y$-axis. $x = -y^2 + 4y$ (p.461: 12)

Revolving about a line that is NOT the $x$ or $y$ axis.

Example 4: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the line $y = 1$: $y = 3 - \frac{x^2}{2}$

Example 5: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the line $x = 2$: $y = 2x^2$ (p.461: 14d)
The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative **washer**.

Volume of the washer = \( \pi(R^2 - r^2)w \)

Where \( R \) = the outer radius and \( r \) = inner radius

If rotated around a horizontal axis, then

Volume of solid = \( \pi \int_a^b ([R(x)]^2 - [r(x)]^2) \, dx \)

If rotated around a vertical axis, then

Volume of solid = \( \pi \int_c^d ([R(y)]^2 - [r(y)]^2) \, dy \)

Example 1: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by the graphs: \( y = x^2, \ y = \sqrt{x} \), about \( x \)-axis.

(answer is \( \frac{3\pi}{10} \))
Example 2: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by the graphs: \( y = 2x^2, \ y = 0, \ x = 2 \) about the line \( y = 8 \).  (p.461: 14d)

Example 3: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by the graphs: \( y = 2x^2, \ y = 0, \ x = 2 \) about the y-axis.  (p.461: 14c)

For the following problems, (a) Find the outer radius, R, and the inner radius, r, (b) find the limits of integration, and (c) set up the integral that gives the volume of the solid bounded by the given functions.

For problems 1-3, also find the volume.

**Homework**

1. \( y = x^2, \ y = x, \) about the y-axis.
2. \( y = x^2, \ y = x, \) about line \( x = -1 \).
3. \( y = x^2, \ y = x, \) about line \( y = 3 \).
4. \( y = (x - 1)^2 + 1, \ y = 1, \ x = 0, \ x = 1 \) about the x-axis.
5. \( y = (x - 1)^2 + 1, \ y = 1, \ x = 0, x = 1 \) about line \( x = -1 \).
6. \( y = (x - 1)^2 + 1, \ y = 1, \ x = 0, x = 1 \) about line \( x = 2 \).

7.2 p. 461: 5-15 odd, 19, 21, 29, 31
Volume: The Shell Method

Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by the graphs: \( y = 4x - x^2, \ y = 0, \) about \( y \)-axis.

Perhaps, we need an alternate method. The shell method will help us out of this messy algebra. In the shell method, a strip that is parallel to the axis of rotation is taken so that when it is rotated, it forms a cylindrical shell. Animation: [https://youtu.be/JrRniVSW9tg](https://youtu.be/JrRniVSW9tg)

Volume of the shell \( \approx 2\pi rh \times \text{thickness} \). Where \( r \) = the distance from the axis of rotation. Note: The representative rectangle will ALWAYS be parallel to the axis of rotation.
There are two general formulas for finding the volume by the shell method.

**Vertical Axis**

\[
V = 2\pi \int_a^b r(x) \cdot h(x) \, dx
\]

**Horizontal Axis**

\[
V = 2\pi \int_c^d r(y) \cdot h(y) \, dy
\]

Example 1: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by the graphs: \( y = 4x - x^2 \), \( y = 0 \), about \( y \)-axis.

Answer is \( \frac{128\pi}{3} \)

Example 2: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by the graphs: \( x = (y - 2)^2 \), \( y = x \), about the line \( y = -1 \).

Answer is \( \frac{63\pi}{2} \)
Example 3: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by the graphs: \( y = \sqrt{x + 2} \), \( y = x \), \( y = 0 \) about the \( x \)-axis. (p.470: 22)

Answer is \( \frac{16\pi}{3} \)

Example 4: Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by the graphs: \( y = x^2 \), \( y = x^3 \), \( y = 0 \) about the \( y \)-axis. Use the disk method and then use the shell method. Which method is easier? Did you get the same result?
Finding the length of a curve:


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**Definition of Arc Length**

Let the function \( y = f(x) \) represent a smooth curve on the interval \([a, b]\). The **arc length** of \( f \) between \( a \) and \( b \) is

\[
s = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx.
\]

Similarly, for a smooth curve \( x = g(y) \), the **arc length** of \( g \) between \( c \) and \( d \) is

\[
s = \int_{c}^{d} \sqrt{1 + [g'(y)]^2} \, dy.
\]

**Examples:** Find the length of the curve over the given interval.

1. \( y = 2x^{\frac{3}{2}} + 3, \quad [0,9] \)
2. \( y = \frac{x^3}{6} + \frac{1}{2x}, \quad [1,2] \)

3. \( y = \frac{x^3}{10} + \frac{1}{6x^3}, \quad [2,5] \)
Surface Area

Find the surface area obtained by revolving a given curve about an indicated axis. Recall from geometry how to find the lateral surface area of a cylinder.

So, for the purposes of the derivation of the formula, let’s look at rotating the continuous function \( y = f(x) \) in the interval \([a, b]\) about the \(x\)-axis. Below is a sketch of a function and the solid of revolution we get by rotating the function about the \(x\)-axis.

Now, rotate the approximations about the \(x\)-axis and we get the following solid.

The approximation on each interval gives a distinct portion of the solid. Each of these portions is called frustums and the surface area a frustum is given by:

\[
SA = \text{Circumference} \times \Delta s
\]

If we make the distances between the points small, then we can say the surface area is

\[
\Delta SA \approx (\text{Circumference})\Delta s.
\]
As \( \Delta s \to 0 \), then we have the differential: \( dSA = (\text{Circumference})ds \)

\[
SA = \int (\text{circumference})(\text{ArcLength})
\]

\[
SA = \int 2\pi rds
\]

\[
ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}
\]

[Image of a graph with a curve and a differential element labeled \( ds \) and \( r \)]

http://demonstrations.wolfram.com/SurfaceAreaOfASolidOfRevolution/

**Review of algebra**

Simplify the following

\[
\sqrt{2 + x} \sqrt{1 + \frac{x^2}{2 + x}}
\]

\[
\left( x^{\frac{1}{3}} \right)^2 = \ldots
\]

\[
x \sqrt{1 + \frac{1}{x^{\frac{2}{3}}}}
\]
Examples

http://www.shodor.org/interactivate/activities/FunctionRevolution/

Find the surface area obtained by revolving the given curve about the indicated axis TWO ways—one using $dx$ and one using $dy$:

$$y = 2\sqrt{x}, \quad 1 \leq x \leq 4 \text{ about the } x\text{-axis}$$
Find the surface area obtained by revolving the given curve about the indicated axis:

\[ x = y^2 - 1, \quad 1 \leq y \leq 4 \] about the \( x \)-axis
Find the surface area obtained by revolving the given curve about the indicated axis TWO ways—one using $dx$ and one using $dy$:

$$x = \sqrt{2y - 1}, \quad \frac{5}{8} \leq y \leq 1 \text{ about the } y\text{-axis}$$
Homework problems:

P. 481: 7,9,11,39,43,45,47 (Hint: refer to notes on arc length) and the following:

Find the arc length of the graph of the function over the indicated interval. (Exact values!)

1. \( y = \frac{x^3}{12} + \frac{1}{x}, \quad 1 \leq x \leq 4 \)

2. \( y = \frac{x^{\frac{3}{2}}}{3} - x^{\frac{1}{2}}, \quad 1 \leq x \leq 9 \)

3. \( y = \frac{x^4}{4} + \frac{1}{8x^2}, \quad 1 \leq x \leq 2 \)

Set up the simplified definite integral that would calculate the area of the surface generated by revolving the curve about the given axis. Evaluate the integral on #4 and #5 (Exact values!)

4. \( y = \sqrt{2x - x^2}, \quad \frac{1}{2} \leq x \leq \frac{3}{2} \) about the x-axis

5. \( x = \frac{y^3}{3}, \quad 0 \leq y \leq 1 \) about the y-axis

6. \( x = 2\sqrt{4 - y}, \quad 0 \leq y \leq \frac{15}{4} \) about the y-axis

7. \( y = 1 - x^2, \quad 0 \leq x \leq 1 \) about the y-axis

8. \( x = 1 + 2y^2, \quad 1 \leq y \leq 2 \) about the x-axis

9. \( x = \frac{1}{3}(y^2 + 2)^{\frac{3}{2}}, \quad 1 \leq y \leq 2 \) about the x-axis
Physics: Work Done by a Force

Definition:
If an object is moved a distance, $d$, in the direction of an applied constant force, $F$, then the work, $W$, done by the force is defined as $W = Fd$.

However, most forces are not constant (varies-compressing a spring) and will depend upon where exactly the force is acting. If the force is given by $F(x)$, then the work done by the force along the x-axis from $x = a$ to $x = b$ is given by,

$$ W = \int_{a}^{b} F(x) dx $$

Hooke’s Law:
The force $F$ required to compress or stretch a spring is proportional to the distance $d$ that the spring is compressed or stretched form its original length. That is,

$$ F = kd $$

where $k$ is the spring constant.

Body Weight:
The weight of a body varies inversely as the square of its distance from the center of the Earth. The force $F(x)$ exerted by gravity is

$$ F(x) = \frac{C}{x^2} $$

where the radius of the Earth is approximately 4000 miles

Example 1: A force of 250 newtons stretches a spring 30 centimeters. How much work is done in stretching the spring from 20 centimeters to 50 centimeters?
Example 2: Seven and one-half foot-pounds of work is required to compress a spring 2 inches from its natural length. Find the work required to compress the spring an additional one-half inch.

Example 3: A lunar module weighs 12 tons on the surface of Earth. How much work is done in propelling the module from the surface of the moon to a height of 50 miles? Consider the radius of the moon to be 1100 miles and its force of gravity to be one-sixth that of the Earth.
Moments and Center of Mass

Archimedes used "moving power" to describe the effect of a lever in moving a mass on the other end. He is famous for the quote "Give me a lever long enough and a fulcrum on which to place it, and I shall move the world." We will define this idea as "moment"

We will look at moments about
- A point—1 dimension
- A line—2 dimensions
- A plane—3 dimensions

**Moment = (mass) * (distance)**

I. One-Dimensional System

The measure of the tendency of this system to rotate about the origin is the **moment about the origin**. 

\[ M_0 = m_1x_1 + m_2x_2 + \cdots + m_nx_n \]

\[ M_0 = \text{moment about the origin}. \text{ When } M_0 = 0, \text{ the system is said to be in equilibrium.} \]

**Center of Mass**: point where the system is in equilibrium, denoted by \( \bar{x} \)

\[ \bar{x} = \frac{M_0}{m}, \text{ where } m = m_1 + m_2 + \cdots + m_n \text{ is the total mass of the system.} \]

Example 1: Find the moment about the origin and center of mass of a system of four objects with masses 10 g, 45 g, 32 g, and 24 g that are located at the points \(-4, 1, 3, \text{ and } 8\), respectively, on the \( x \)-axis.

Example 2: If a 20 pound child and a 60 pound child sit on the ends of a seesaw that is 5 feet long, where must the fulcrum be located so that the beam will balance?
II. Two-Dimensional System

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

1. The **moment about the y-axis** is \[ M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n. \]

2. The **moment about the x-axis** is \[ M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n. \]

3. The **center of mass** $(\bar{x}, \bar{y})$ (or center of gravity) is

\[
\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}
\]

where $m = m_1 + m_2 + \cdots + m_n$

is the total mass of the system.

Example 3: Find the center of mass of the given system of point masses.

<table>
<thead>
<tr>
<th>$m_i$</th>
<th>3</th>
<th>4</th>
<th>2</th>
<th>1</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x_i, y_i)$</td>
<td>$(-2, -3)$</td>
<td>$(5, 5)$</td>
<td>$(7, 1)$</td>
<td>$(0, 0)$</td>
<td>$(-3, 0)$</td>
</tr>
</tbody>
</table>

III. Three-Dimensional System-Using geometry

Mass = (density) * (Area) or Mass = ($\rho$) * (Area).

A planar lamina (a thin plate) with uniform density $\rho$. If we take the density to equal 1, then the mass is numerically equal to the area.

Example 4: Find the center of mass of the region shown.
III. Three-Dimensional System-Irregular Laminas

\[
\bar{x} = x \quad \text{and} \quad \bar{y} = \frac{1}{2} (f(x) + g(x))
\]

<table>
<thead>
<tr>
<th>Representative Thin Strip (rectangle):</th>
<th>Full Region</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Area of representative rectangle:</td>
<td>1. Area of region:</td>
</tr>
<tr>
<td>( \Delta A = (f(x) - g(x))\Delta x )</td>
<td>( A = \int_a^b (f(x) - g(x))dx )</td>
</tr>
<tr>
<td>(Height*width)</td>
<td>2. The moment about the ( y )-axis is</td>
</tr>
<tr>
<td>2. The moment about the ( y )-axis is</td>
<td>( M_y = \int_a^b x(f(x) - g(x))dx )</td>
</tr>
<tr>
<td>( M_y = \bar{x}\Delta A = x(f(x) - g(x))\Delta x )</td>
<td>3. The moment about the ( x )-axis is</td>
</tr>
<tr>
<td>3. The moment about the ( x )-axis is</td>
<td>( M_x = \frac{1}{2} \int_a^b ([f(x)]^2 - [g(x)]^2)dx )</td>
</tr>
<tr>
<td>( M_x = \bar{y}\Delta A = \frac{1}{2}(f(x) + g(x))(f(x) - g(x))\Delta x )</td>
<td>4. The center of mass ((\bar{x}, \bar{y})) (or center of gravity) is</td>
</tr>
<tr>
<td>( \bar{y} = \frac{1}{2}(f(x) + g(x)) )</td>
<td>( \bar{x} = \frac{M_y}{A} ) and ( \bar{y} = \frac{M_x}{A} )</td>
</tr>
</tbody>
</table>

Example 5: Find the center of mass of a lamina of uniform density \( \rho \) covering the region bounded by the parabola \( y = 4 - x^2 \) and \( y = 0 \). Let \( \rho = 1 \) (Hint: symmetry)
Example 6: Find the center of mass of a lamina of uniform density $\rho$ covering the region bounded by the parabola $y = x^4$ and $y = x$. Let $\rho = 1$
Example 7: Find the center of mass of a lamina of uniform density $\rho$ covering the region bounded by the parabola $x = y + 2$ and $x = y^2$. Let $\rho = 1$.
Review Basic Inverse Trig Derivatives:

\[
\frac{d}{dx} \left( \arcsin u \right) = \frac{u'}{\sqrt{1-u^2}}
\]

\[
\frac{d}{dx} \left( \arctan u \right) = \frac{u'}{1+u^2}
\]

\[
\frac{d}{dx} \left( \arccos u \right) = -\frac{u'}{\sqrt{1-u^2}}
\]

\[
\frac{d}{dx} \left( \arccot u \right) = -\frac{u'}{1+u^2}
\]

\[
\frac{d}{dx} \left( \arccsc u \right) = \frac{-u'}{|u|\sqrt{u^2-1}}
\]

Review Basic Inverse Trig Integration:

\[
\int \frac{du}{\sqrt{a^2-u^2}} = \quad \int \frac{du}{a^2+u^2} = \quad \int \frac{du}{u\sqrt{u^2-a^2}} =
\]

Find the indefinite integral:

\[
\int \frac{1}{2+9x^2} \, dx 
\]

\[
\int \frac{1}{\sqrt{25-36x^2}} \, dx 
\]

\[
\int \frac{1}{x\sqrt{4x^2-9}} \, dx 
\]
\[ \int \frac{x}{x^4 + 16} \, dx \]

\[ \int \frac{\cos^{-1} x}{\sqrt{1 - x^2}} \, dx \]

\[ \int \frac{1}{\sqrt{x(1 + x)}} \, dx \]

\[ \int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{1 + \cos^2 x} \, dx \]

**Completing the Square**

\[ x^2 - 6x + 10 \]

\[ 2x - x^2 \]
\[\int \frac{1}{\sqrt{8 - 6x - x^2}} \, dx\]

\[\int \frac{6}{\sqrt{3 + 4x - 4x^2}} \, dx\]

\[\int \frac{2x - 5}{x^2 + 2x + 2} \, dx\]
**Improper Fractions:**

The degree of the numerator is greater than or equal to the degree of the denominator. You must DIVIDE to change to a "mixed" fraction before you integrate.

\[ \int \frac{x^4 - 1}{x^2 + 1} dx \]

\[ \int \frac{x^2}{x^2 + 1} dx \]

\[ \int \frac{4x^2 - 7}{2x + 3} dx \]  
(Hint: Remember \( \int \frac{1}{u} du = \ln |u| + C \))

**Separating Fractions:**

You CANNOT separate denominators.

\[ \int \frac{x^4 - x}{x^2 + 1} dx \neq \int \frac{x^4}{x^2} - \frac{x}{1} dx \]

\[ \int \frac{2 - 8x}{1 + 4x^2} dx \]
Review Basic Trig Integration:
\[ \int \sin(u) \, du = -\cos(u) + C \]
\[ \int \cos(u) \, du = \sin(u) + C \]
\[ \int \sec^2(u) \, du = \tan(u) + C \]
\[ \int \csc^2(u) \, du = -\cot(u) + C \]
What about
\[ \int \tan(u) \, du \]
\[ \int \cot(u) \, du \]
\[ \int \sec(u) \, du \]
\[ \int \csc(u) \, du \]

Multiplying by a Form of One:
\[ \int \sec(u) \, du \]
\[ \int \csc(u) \, du \]

\[ \int \frac{dx}{1 + \cos x} \]
\[ \int \frac{dx}{1 - \csc x} \]
Trig Identities:

\[
\begin{align*}
\cos^2 \theta + \sin^2 \theta &= 1 \\
1 + \tan^2 \theta &= \sec^2 \theta \\
\cot^2 \theta + 1 &= \csc^2 \theta \\
\sin 2\theta &= 2 \sin \theta \cos \theta \\
\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\
&= 2 \cos^2 \theta - 1 \\
&= 1 - 2 \sin^2 \theta
\end{align*}
\]

\[
\int (\csc x - \tan x)^2 \, dx
\]

Review Other Bases Integration:

\[
\int a^u \, du = \\
\int 8^{-x} \, dx
\]

Log Rule:

What about \( \int \ln(u) \, du \)?

\[
\int \cot(x) \ln(\sin x) \, dx
\]

"Math Magic Tricks"

\[
\int \frac{1}{1 + e^x} \, dx
\]
Basic Integrations Formulas (Thomas' CALCULUS Media Upgrade 11th edition)

Basic Substitutions

1. \( \int \frac{16x}{\sqrt{8x^2 + 1}} \, dx \)  
3. \( \int 3\sqrt{\sin x \cos x} \, dx \)  
5. \( \int \frac{16x}{8x^2 + 2} \, dx \)  
7. \( \int \frac{dx}{\sqrt{x} (\sqrt{x} + 1)} \)

9. \( \int \cot(3 - 7x) \, dx \)  
11. \( \int e^\theta \csc(e^\theta + 1) \, d\theta \)  
13. \( \int \sec \frac{t}{3} \, dt \)  
15. \( \int \csc(x - \pi) \, dx \)

17. \( \int_0^{\text{sin}^2} 2x \, e^x \, dx \)  
19. \( \int e^{\tan x} \sec^2 x \, dx \)  
21. \( \int 3x^2 \, dx \)  
23. \( \int \frac{2\sqrt{w}}{2\sqrt{w}} \, dw \)

25. \( \int_0^{\sqrt{5}} \frac{dx}{4 + 9x^2} \)  
27. \( \int_0^{\frac{\sqrt{3}}{3}} \frac{dx}{\sqrt{1 - 9x^2}} \)  
29. \( \int \frac{2x \, dx}{\sqrt{1 - x^4}} \)  
30. \( \int (\tan x)[\ln(\cos x)] \, dx \)

31. \( \int \frac{6 \, dx}{x\sqrt{25x^2 - 1}} \)  
33. \( \int \frac{dx}{e^x + e^{-x}} \)  
35. \( \int_1^{e^{\frac{x}{2}}} \frac{dx}{x \cos(\ln x)} \)  
36. \( \int \frac{\ln(x^2)}{x} \, dx \)

Completing the Square

37. \( \int_{\frac{2}{2}}^{\frac{2}{2}} \frac{8 \, dx}{x^2 - 2x + 2} \)  
39. \( \int \frac{dx}{\sqrt{x^2 - 4x + 3}} \)  
41. \( \int \frac{dx}{(x + 1)\sqrt{x^2 + 2x}} \)

Trigonometric Identities

43. \( \int (\sec x + \cot x)^2 \, dx \)

Improper Fractions

47. \( \int \frac{x}{x + 1} \, dx \)  
49. \( \int_{\sqrt{2}}^{\frac{2\sqrt{3}}{2}} \frac{2x^3}{x^2 - 1} \, dx \)  
51. \( \int \frac{4x^3 - x^2 + 16x}{x^2 + 4} \, dx \)

Separating Fractions

53. \( \int \frac{1 - x}{\sqrt{1 - x^2}} \, dx \)  
55. \( \int_0^{\frac{\pi}{4}} \frac{1 + \sin x}{\cos^2 x} \, dx \)

Multiplying by a Form of 1

57. \( \int \frac{1}{1 + \sin x} \, dx \)  
59. \( \int \frac{1}{\sec x + \tan x} \, d\theta \)  
61. \( \int \frac{1}{1 - \sec x} \, dx \)

Trigonometric Powers

83. \( \int \cos^3 \theta \, d\theta \) (Hint: \( \cos^2 \theta = 1 - \sin^2 \theta \))
Answers

1. \( \sqrt{8x^2} + 1 + C \)
2. \( (\sin x)^{3/2} + C \)
3. \( \ln 10 - \ln 2 = \ln 5 \)
4. \( 2 \ln(\sqrt{x} + 1) + C \)
5. \( -\frac{1}{7} \ln |\sin(3 - 7x)| + C \)
6. \( -\ln(csc(e^x + 1) + cot(e^x + 1)) + C \)
7. \( 3 \ln \left| \sec \left( \frac{x}{3} \right) + \tan \left( \frac{x}{3} \right) \right| + C \)
8. \( e^{\ln^2 x} - e^0 = 2 - 1 = 1 \)
9. \( e^{\tan^2 x} + C \)
10. \( \frac{3(x+1)}{\ln 3} + C \)
11. \( 2\sqrt{w} + C \)
12. \( \frac{\pi}{18} \)
13. \( \frac{\pi}{18} \)
14. \( \arcsin(x^2) + C \)
15. \( \frac{-[\ln(\cos x)]^2}{2} + C \)
16. \( 6 \sec^{-1} |5x| + C \)
17. \( \arctan(x^3) + C \)
18. \( \ln(2 + \sqrt{3}) \)
19. \( (\ln x)^2 + C \) or \( 2\pi \)
20. \( \sin^{-1}(x - 2) + C \)
21. \( \sec^{-1} |x + 1| + C \)
22. \( \tan x - 2 \ln|\csc x + \cot x| - \cot x - x + C \)
23. \( x - \ln |x + 1| + C \)
24. \( 7 + \ln 8 \)
25. \( 2x^2 - x + 2 \tan^{-1} \left( \frac{x}{2} \right) + C \)
26. \( \sin^{-1} x + \sqrt{1 - x^2} + C \)
27. \( \sqrt{2} \)
28. \( \tan x - \sec x + C \)
29. \( \ln |1 + \sin \theta| + C \)
30. \( x + \cot x + \csc x + C \)
31. \( \sin \theta - \frac{1}{3} \sin^3 \theta + C \)
Integration by Parts

If you are given a function \( y = f(x)g(x) \) and asked to take the derivative, you would use the product rule and get:

\[
\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)
\]

If we let \( u = f(x) \) and \( v = g(x) \), then \( du = f'(x)dx \) and \( dv = g'(x)dx \)

We can write the product rule as:

\[
\frac{d}{dx}(uv) = u dv + v du
\]

In terms of indefinite integrals, this equation becomes:

\[
\int \frac{d}{dx}(uv)dx = \int [u dv + v du]
\]

We can separate the right hand side:

\[
\int \frac{d}{dx}(uv)dx = \int u dv + \int v du
\]

We can rearrange and get:

\[
\int \frac{d}{dx}(uv)dx - \int v du = \int u dv
\]

We can replace and get:

\[
uv - \int v du = \int u dv
\]

**Integration by Parts Formula:**

\[
\int u dv = uv - \int v du
\]

\[
\int \frac{5x}{e^{2x}}dx
\]
\[ \int x^4 \ln(3x) \, dx \]

\[ \int \ln x \, dx \]

\[ \int 4x \sec^2(2x) \, dx \]

\[ \int 4 \arccos x \, dx \]
\[ \int x^2 \sin x \, dx \]

**Tabular Method**

Use this method if one part can be "derived" down to zero and other part can be integrated repeatedly easily. Useful for: powers of x multiplied to e's, sines or cosines

\[ \int x^3 e^{3x} \, dx \]

\[ \int x^3 \cos(2x) \, dx \]

***\[ \int x^5 \ln(3x) \, dx \] Can you use the tabular method for this problem?***
Are we going in circles?

\[ \int e^{4x} \cos(2x) \, dx \]
Trigonometric Integrals

Pythagorean Identities:
\[
\cos^2 \theta + \sin^2 \theta = 1 \\
\cos^2 \theta = 1 - \sin^2 \theta \\
\sin^2 \theta = 1 - \cos^2 \theta \\
\sec^2 \theta = 1 + \tan^2 \theta \\
\tan^2 \theta = \sec^2 \theta - 1
\]
\[
\cos 2\theta = 2 \cos^2 \theta - 1 \\
\cos 2\theta = 1 - 2 \sin^2 \theta \\
\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \\
\sin 2\theta = 2 \sin \theta \cos \theta
\]

Trig Review:
\[
\frac{d}{dx} [\sin x] = \cos x \\
\int \cos(x) \, dx = \sin(x) + C
\]
\[
\frac{d}{dx} [\cos x] = -\sin x \\
\int \sin(x) \, dx = -\cos(x) + C
\]
\[
\frac{d}{dx} [\tan x] = \sec^2 x \\
\int \sec^2(x) \, dx = \tan(x) + C
\]
\[
\frac{d}{dx} [\sec x] = \sec x \tan x \\
\int \sec(x) \tan(x) \, dx = \sec(x) + C
\]
\[
\int \sec(x) \, dx = \ln |\sec(x) + \tan(x)| + C \\
\int \tan(x) \, dx = -\ln |\cos x| + C \text{ or } \ln |\sec x| + C
\]

Powers of Sines and Cosines
\[
\int \sin^3(x) \cos(x) \, dx \\
\int \sin^4(x) \cos(x) \, dx
\]
\[
\int \sin^7(x) \cos(x) \, dx \\
\int \sqrt{\sin(x)} \cos(x) \, dx
\]

Conclusion:
\[ \int (1 - \sin^3(4x) + \sin^5(4x)) \cos(4x) dx \]

\[ \int \cos^3(x) \sin(x) dx \]

**Conclusion:**

\[ \int (\cos^2(2x) - \cos^3(2x)) \sin(2x) dx \]

\[ \int \sin^2(x) \cos^3(x) dx \]

\[ \int \sin^5(x) \cos^4(x) dx \]
Even Powers of Sines and Cosines

\[ \int \cos^2(x) \, dx \quad \int \sin^2(4x) \, dx \]

\[ \int \sin^2(x) \cos^2(x) \, dx \]

POWERS OF TANGENT AND SECANT

\[ \int \sec^6 x (\sec x \tan x) \, dx \quad \int \sec^2 x \tan^3 x \, dx \]

Conclusion:
\int \tan^2(x) \sec^4(x) \, dx

\int \tan^3(x) \sec^3(x) \, dx

\int \tan^4(x) \, dx \quad \int \sec^3(x) \, dx
Trig Substitution

Trig substitutions occur when we replace the variable of integration by a trig function. The most common substitutions are:

\[
\begin{align*}
\sin \theta &= \frac{u}{a} & \tan \theta &= \frac{u}{a} & \sec \theta &= \frac{u}{a} \\
& \text{Or} \\
& a \sin \theta = u & a \tan \theta = u & a \sec \theta = u
\end{align*}
\]

1. \[ \int \frac{3}{\sqrt{1 + 9x^2}} \, dx \]

2. \[ \int \sqrt{1 - 9x^2} \, dx \]
3. \[ \int \frac{\sqrt{y^2 - 25}}{y^3} \, dy \]

4. \[ \int \frac{4}{x^2 \sqrt{16 - x^2}} \, dx \]
5. $\int \frac{\sqrt{25x^2 + 4}}{x^4} \, dx$

6. $\int \frac{1}{(x^2 + 5)^{3/2}} \, dx$
Suppose \( f(x) = \frac{g(x)}{h(x)} \) where \( g(x) \) and \( h(x) \) are polynomials and the degree of \( h(x) \) is BIGGER than the degree of \( g(x) \) (if this is not true, you must first do long division), we need to decompose the rational function into simpler rational functions.

In order to integrate a partial fraction problem, you must first find the Partial Fraction Decomposition:

**Case I:** \( h(x) \) is a product of linear factors, none repeating, then the Partial Fraction Decomposition has the form:

\[
\frac{x + 1}{(x - 2)(2x - 11)} = \frac{A}{x - 2} + \frac{B}{2x - 11}
\]

Watch this video: Partial Fractions-Part 1

**Case II:** \( h(x) \) is a product of linear factors, some repeating, then the Partial Fraction Decomposition has the form:

\[
\frac{x + 1}{(x - 1)(x - 2)^2} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{(x - 2)^2}
\]

Watch this video: Partial Fractions-Part 2

**Case III:** \( h(x) \) contains irreducible quadratic factors, none repeating, then the Partial Fraction Decomposition has the form:

\[
\frac{x + 1}{(x - 2)(x^2 + 1)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 1}
\]

**Case IV:** \( h(x) \) contains irreducible quadratic factors, some repeating, then the Partial Fraction Decomposition has the form:

\[
\frac{x + 5}{(x - 2)(x^2 + 4)^2} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 4} + \frac{Dx + E}{(x^2 + 4)^2}
\]

Watch this video for Case III and IV: Partial Fractions-Part 3

Once you find \( A, B, \) etc., then you integrate the result.
Integrate the following:

1. \[ \int \frac{x + 3}{2x^3 - 8x} \, dx \]

2. \[ \int \frac{2x}{2x^2 + 7x + 5} \, dx \]
3. \[ \int \frac{x^3 + 3}{x^2 + x - 2} \, dx \]

4. \[ \int \frac{1}{(x-1)^2(x+4)} \, dx \]
5. \[ \int \frac{3x^2 - 4x + 5}{(x - 1)(x^2 + 1)} \, dx \]

6. \[ \int \frac{8x^2 + 8x + 2}{(4x^2 + 1)^2} \, dx \]
L'Hôpital's Rule

Review of Limits

\[
\begin{align*}
\lim_{x \to 2} \frac{2x + 3}{x - 4} & \quad \lim_{x \to 4} \frac{x^2 - 16}{x - 4} & \quad \lim_{x \to \infty} \frac{4x^2 - 5x}{1 - 3x^2}
\end{align*}
\]

Indeterminate Forms

\[
\begin{array}{cccccc}
0 & \pm\infty & 0 \cdot \infty & 1^\infty & 0^0 & \infty^0
\end{array}
\]

\[
\begin{align*}
\lim_{x \to 2} \frac{2\cos \pi x - 2}{x^2 - 4} & \quad \lim_{x \to 2} \frac{2x + 3}{x^2 - 4}
\end{align*}
\]

L'Hôpital's Rule

Suppose that we have one of the following cases:

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{OR} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \pm\infty
\]

where \(a\) can be any real number, infinity or negative infinity.

In these cases we have:

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

Find the limit:

1. \(\lim_{x \to \infty} \frac{\ln x^4}{x^3}\)  
2. \(\lim_{x \to 0} \frac{\cos x}{1 + \tan x}\)  
3. \(\lim_{x \to \infty} \frac{x^2}{e^x}\)
4. \( \lim_{x \to \infty} \left( x \tan \left( \frac{1}{x} \right) \right) \)

5. \( \lim_{x \to 0^+} \left( \frac{10}{x} - \frac{3}{x^2} \right) \)

6. \( \lim_{x \to \infty} \left( 1 + \frac{1}{x^2} \right)^x \)

7. \( \lim_{x \to 1^+} x^{\frac{1}{x-1}} \)
Improper Integrals

**Type 1**

Limits of integration that involve ±∞

\( f(x) \) is continuous on \([a, \infty)\):
\[
\int_{a}^{\infty} f(x) \, dx = \lim_{c \to \infty} \int_{a}^{c} f(x) \, dx
\]

\( f(x) \) is continuous on \((-\infty, b]\):
\[
\int_{-\infty}^{b} f(x) \, dx = \lim_{c \to -\infty} \int_{c}^{b} f(x) \, dx
\]

\( f(x) \) is continuous on \((-\infty, \infty)\):
\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{c} f(x) \, dx + \int_{c}^{\infty} f(x) \, dx
= \lim_{b \to -\infty} \int_{-\infty}^{b} f(x) \, dx + \lim_{a \to \infty} \int_{a}^{\infty} f(x) \, dx
\]

If the limit exists, then the integral **converges**.
If the limit does not exist, then the integral **diverges**.

1. \( \int_{1}^{\infty} \frac{1}{x^2} \, dx \)

2. \( \int_{-\infty}^{0} \frac{1}{\sqrt{3-x}} \, dx \)
3. \( \int_{-\infty}^{\infty} \frac{4}{16 + x^2} \, dx \)

4. \( \int_{1}^{\infty} \frac{(1-x)}{e^x} \, dx \)

**Type 2**

Vertical asymptotes within the given limits of integration. (Function is undefined)

\( f(x) \) has an infinite discontinuity at \( x = c \), \( c \in (a, b) \)

\[
\int_{a}^{c} f(x) \, dx = \lim_{b \to c} \int_{a}^{b} f(x) \, dx \\
\int_{b}^{c} f(x) \, dx = \lim_{a \to c} \int_{a}^{c} f(x) \, dx \\
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx
\]
5. \[ \int_{0}^{2} \frac{1}{x-2} \, dx \]

6. \[ \int_{3}^{5} \frac{1}{\sqrt{x^2 - 9}} \, dx \]

7. \[ \int_{-1}^{1} \frac{1}{x^2} \, dx = \int_{-1}^{1} x^{-2} \, dx \]

\[ = -\frac{1}{x} \bigg|_{-1}^{1} = (-1 - 1) = -2 \]

Why is this incorrect?

11th ed. P 579: 17,19, 21,23,33-41 odd,45
In this section, we present a new concept which allows us to use functions to study curves which, when plotted in the \(xy\)-plane, neither represent \(y\) as a function of \(x\) nor \(x\) as a function of \(y\).

We can define the \(x\)-coordinate of \(P\) as a function of \(t\) and the \(y\)-coordinate of \(P\) as a (usually, but not necessarily) different function of \(t\). (Traditionally, \(f(t)\) is used for \(x\) and \(g(t)\) is used for \(y\).) The independent variable \(t\) in this case is called a parameter and the system of equations \[ \begin{align*} x &= f(t) \\ y &= g(t) \end{align*} \] is called a system of parametric equations.

For the following, sketch the curve, indicate its orientation, and eliminate the parameter to get an equation involving just \(x\) and \(y\).

**Ex. 1**  \[ \begin{align*} x &= t^2 - 3 \\ y &= 2t - 1 \end{align*} \]
Ex. 2 \[ \begin{aligned} x &= t^{\frac{3}{2}} + 1 \\ y &= \sqrt{t} \end{aligned} \]

Ex. 3 \[ \begin{aligned} x &= 2\cos t \\ y &= 3\sin t \end{aligned} \]
Parametric Equations and Calculus

Parametric Form of the Derivative

If a smooth curve $C$ is given by the equations: $x = f(t)$ and $y = g(t)$,

then the slope of $C$ at $(x, y)$ is

$$\frac{dy}{dx} = \frac{dy/\,dt}{dx/\,dt}, \quad \frac{dx}{dt} \neq 0$$

The second derivative is

$$\frac{d^2 y}{dx^2} = \frac{d}{dt} \left( \frac{dy/\,dt}{dx/\,dt} \right)$$

Tangent Lines

There is a horizontal tangent line if $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

There is a vertical tangent line if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

Examples:

Find $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$. Find the slope and concavity (if possible) at the given value of the parameter.

Find the $(x, y)$ coordinate at given value of $t$.

1. $x = \sqrt{t}, \quad y = 3t - 1, \quad t = 1$
2. \( x = 2\sin(2\pi t), \quad y = \cos(2\pi t), \quad t = -\frac{1}{6} \)

Find all points (if any) of horizontal and vertical tangency to the curve.

3. \( x = t^4 + 2, \quad y = t^3 + t \)

4. \( x = 8\sin t, \quad y = 4\cos t \)
5. Find the equation of the tangent line(s) to the given set of parametric equations at the given value of the parameter. \( x = t^3 - 4t^2, \quad y = t^2 \) when \( t = -2 \)

**Arc Length**

**Definition:** If a curve \( C \) is defined parametrically by \( x = f(t) \) and \( y = g(t) \), \( a \leq t \leq b \), where \( f' \) and \( g' \) are continuous and not simultaneously zero on \([a,b] \), and \( C \) is traversed exactly once as \( t \) increases from \( t = a \) to \( t = b \), then the length of \( C \) is the definite integral:

\[
L = \int_{a}^{b} \sqrt{\left( f'(t) \right)^2 + \left( g'(t) \right)^2} \, dt,
\]

\[
L = \int_{a}^{b} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt
\]

Find the arc length of the curve on the given interval:

6. \( x = 6t^2, \quad y = 2t^3, \quad 1 \leq t \leq 4 \)
Area of a Surface of Revolution

If a smooth curve \( x = f(t) \) and \( y = g(t) \), \( a \leq t \leq b \), is traversed exactly once as \( t \) increases from \( t = a \) to \( t = b \), then areas of the surfaces generated by revolving the curve about the coordinate axes are as follows:

1. **Revolution about the \( x \)-axis \((y \geq 0)\):**

\[
S = \int_{a}^{b} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

2. **Revolution about the \( y \)-axis \((x \geq 0)\):**

\[
S = \int_{a}^{b} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

Find the area of the surface generated by revolving the curve about the given axis:

7. \( x = \frac{t^2}{2}, \ y = 2t, \ 0 \leq t \leq \sqrt{5} \) about the \( x \)-axis

8. \( x = \cos^3 \theta, \ y = \sin^3 \theta, \ 0 \leq \theta \leq \frac{\pi}{2} \) about the \( y \)-axis

11\textsuperscript{th} ed. P. 715:9,11,13,15,33,35,37,39,49,55,63,65 and Worksheet (Parametric-Tangents)
1. \[ x = \cos t + t \sin t, \quad 0 \leq t < 2\pi \]
\[ y = \sin t - t \cos t \]

2. \[ x = 1 - t \]
\[ y = t^3 - 3t \quad -\infty \leq t < \infty \]

3. \[ x = t^2 - t + 2 \]
\[ y = t^3 - 3t \quad -\infty \leq t < \infty \]

4. \[ x = 4 + 2 \cos t \]
\[ y = -1 + \sin t \quad 0 \leq t < 2\pi \]

Find the equation of the tangent line(s) to the given set of parametric equations at the given value of the parameter.

5. \[ x = 2 \cos(3t) - 4 \sin(3t), \quad y = 3 \tan(6t) \quad \text{when} \quad t = \frac{\pi}{2} \]

6. \[ x = t^3 - 2t - 11, \quad y = t(t - 4)^3 - 3t^2(t - 4)^2 + 7 \quad \text{when} \quad t = -2 \]

7. \[ x = t^3 + \cos(\pi t), \quad y = 4t + \sin(2t + 6) \quad \text{when} \quad t = -3 \]

8. \[ x = t^3 + 2t - 1, \quad y = t^3 + 7t^2 + 8t \quad \text{when} \quad t = 1 \]
Solutions:

<table>
<thead>
<tr>
<th></th>
<th>Horizontal:</th>
<th>Vertical:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( t = \pi ) ((-1, \pi))</td>
<td>( t = \frac{\pi}{2} \left( \frac{\pi}{2}, 1 \right) ) ( t = \frac{3\pi}{2} \left( -\frac{3\pi}{2}, -1 \right) )</td>
</tr>
<tr>
<td></td>
<td>Vertical: ( t = \frac{\pi}{2} \left( \frac{\pi}{2}, 1 \right) ) ( t = \frac{3\pi}{2} \left( -\frac{3\pi}{2}, -1 \right) )</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>( t = -1 ) ((2,2))</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>( t = 1 ) ((0,-2))</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>( t = -1 ) ((4,2))</td>
<td>( t = \frac{1}{2} \left( \frac{3}{4}, \frac{-11}{8} \right) )</td>
</tr>
<tr>
<td></td>
<td>( t = 1 ) ((2,-2))</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>( t = \frac{\pi}{2} ) ((4,0))</td>
<td>( t = \frac{3\pi}{2} ) ((4,-2))</td>
</tr>
<tr>
<td></td>
<td>Vertical: ( t = 0 ) ((6,-1)) ( t = \pi ) ((2,-1))</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>( y = 3x - 12 )</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>( y = -24x - 65 )</td>
<td></td>
</tr>
<tr>
<td>7.</td>
<td>( y = \frac{2}{9}x - \frac{52}{9} )</td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td>( y = \frac{25}{4}x + \frac{7}{2} )</td>
<td></td>
</tr>
</tbody>
</table>
Polar Graphs

How to plot polar coordinates: \((r, \theta)\)

The origin is now called the pole. \(r = \) a directed distance from \(O\) to \(P\).

\(\theta = \) directed angle, counterclockwise from polar axis to \(OP\)

Special Polar Graphs

Circles: \(r = a\)

\[ r = 3 \]

Use Graphing Calculator

For TI-83: MODE is Pol, Deg. or Rad,

WINDOW must correspond with Deg. or Rad.

Graph the following with calculator: \(r = 3 + 3\cos \theta\), \(r = 3 + 4\sin \theta\), \(r = 5 – 3\sin \theta\)
**Limaçons**  \( r = a \pm b \cos \theta \)  \( r = a \pm b \sin \theta \)

\[
\begin{align*}
    r &= 1 + 3 \cos \theta \\
    r &= 2 - 2 \sin \theta \\
    r &= 3 + 2 \cos \theta \\
    r &= 4 + 2 \sin \theta
\end{align*}
\]

Limacon:  [https://youtu.be/MT1j6vdX2aI](https://youtu.be/MT1j6vdX2aI)

**Lemniscate (a figure 8)**:  \( r^2 = a^2 \cos(2\theta) \)  \( r^2 = a^2 \sin(2\theta) \)

\[
\begin{align*}
    r^2 &= 25 \cos(2\theta) \\
    r^2 &= 16 \sin(2\theta)
\end{align*}
\]

Lemniscate:  [https://youtu.be/tvPsjmcN2Yg](https://youtu.be/tvPsjmcN2Yg)

**Rose Curves**:  \( r = a \cos(n\theta) \)  \( r = a \sin(n\theta) \)

\[
\begin{align*}
    r &= 3 \cos(3\theta) \\
    r &= 3 \cos(2\theta) \\
    r &= 3 \sin(5\theta) \\
    r &= 3 \sin(2\theta)
\end{align*}
\]

Rose curve:  [https://youtu.be/tvPsjmcN2Yg](https://youtu.be/tvPsjmcN2Yg)
Slope and Tangent Lines

Given that $x = r \cos \theta$, $y = r \sin \theta$ and $r = f(\theta)$, find $\frac{dy}{dx}$.

1. Find $\frac{dy}{dx}$ and the slopes of the tangent lines at the given values of $\theta$.

   $r = -1 + \sin \theta$, $\theta = 0$ and $\theta = \pi$
2. Determine the equation of the tangent line to \( r = 3 + 8\sin \theta \) at \( \theta = \frac{\pi}{6} \)

**Tangent Lines**

There is a horizontal tangent line if \( \frac{dy}{d\theta} = 0 \) and \( \frac{dx}{d\theta} \neq 0 \)

There is a vertical tangent line if \( \frac{dx}{d\theta} = 0 \) and \( \frac{dy}{d\theta} \neq 0 \)
3. Find all the points of vertical and horizontal tangency: \( r = -1 + \sin \theta \)

https://www.youtube.com/watch?v=4SpQ9iOxd_E

4. Review quadratic formula. Solve: \( 4 \cos^2 \theta + \cos \theta - 2 = 0 \) within interval \([0, 2\pi]\)
Tangents at the Pole

There is a tangent line at the pole if $f(\theta) = 0$ and $f'(\theta) \neq 0$, the line is: $\theta = \alpha$

4. Find all the tangents at the pole: $r = 2\cos 3\theta$
A. Find all the points of vertical and horizontal tangency: \( r = 1 + \cos \theta \)

B. Determine the equation of the tangent line to \( r = \sin(4\theta)\cos \theta \) at \( \theta = \frac{\pi}{6} \)

C. Determine the equation of the tangent line to \( r = \theta \sin(3\theta) \) at \( \theta = \frac{\pi}{2} \)

D. Determine the equation of the tangent line to \( r = 1 + 2\cos \theta \) at \( \theta = \frac{\pi}{4} \)

E. Find all the points of vertical and horizontal tangency: \( r = 2 + 4\cos \theta \)

(you will need to use the quadratic formula)

Answers:

1. Horizontal: \((r, \theta) = \left( \frac{3}{2}, \frac{\pi}{3} \right), \left( \frac{3}{2}, \frac{5\pi}{3} \right)\)

2. \( y = \frac{1}{3\sqrt{3}} x + \frac{1}{4} \)

3. \( y = \frac{-2}{\pi} x - \frac{\pi}{2} \)

4. \( y = \frac{1}{7}(1 - 2\sqrt{2}) \left( x - \frac{1 + \sqrt{2}}{\sqrt{2}} \right) + \frac{1 + \sqrt{2}}{\sqrt{2}} \)

5. Bonus
Area and Arc Length of Polar Curves

If we have a circle of radius $r$, and select a sector of angle $\Delta \theta$, where $\beta - \alpha = \Delta \theta$, then the area of that sector is

$$A = \left(\pi r^2\right) \left(\frac{\Delta \theta}{2\pi}\right)$$

Now suppose that we have a curve in polar coordinates. As we did with rectangular coordinates, we take small increments of $\theta$, which we will denote as $\Delta \theta$.

Since $\Delta \theta$ is infinitesimally small, the wedge shaped has an area of $dA = \frac{1}{2} r^2 d\theta$

If we partition the interval $[\alpha, \beta]$ into $n$ equal subintervals:

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \cdots < \theta_{n-1} < \theta_n = \beta$$

Then approximate the area of the region by the sum of the area of the sectors.

Then the approximate area is: $A \approx \sum_{i=1}^{n} \frac{1}{2} \Delta \theta [f(\theta_i)]^2$

Taking the limit as $n \to \infty$,

$$A = \lim_{n \to \infty} \frac{1}{2} \sum_{i=1}^{n} [f(\theta_i)]^2 \Delta \theta$$

$$= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$$

$$\text{Area} = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$
Example #1: Find the area inside the curve: \( r = 1 + \cos \theta \)

Example #2: Find area of one petal of rose curve: \( r = 3\sin 2\theta \)
Example#3: Find the area shared by \( r = 2(1 + \sin \theta) \) and \( r = 2 \)

Example#4: Area outside \( r = 1 - \cos \theta \) and inside \( r = 1 \)
Example#5: Set-up the integral to find the area shared by $r = 2(1 + \sin \theta)$ and $r = 1$

Example#6: Find the area of the inner loop of $r = 2 - 4 \cos \theta$. 
Arc Length in Polar Form

Consider a polar arc: \( r = f(\theta), \ \alpha \leq \theta \leq \beta \) and \( f'(\theta) \) is continuous on interval \( \alpha \leq \theta \leq \beta \).

Remember in parametric form the

\[
\text{Arc Length} = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

Now, we are in polar coordinates. We change the variable \( t \) into \( \theta \):

\[
\text{Arc Length} = \int_{a}^{b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta
\]

We can simplify this by remembering that: \( x = r \cos \theta \) and \( y = r \sin \theta \)
Example: Find the length of the curve over the given interval: \( r = 1 + \cos \theta \quad 0 \leq \theta \leq \pi \)

10.5 11th ed. p.735: 7,9,15,19, 21, 23, 39,41, 43, and Find the area of the following regions.

A. Inside \( r^2 = 6\cos 2\theta \) and outside \( r = \sqrt{3} \) \hspace{2cm} Answer: \( 3\sqrt{3} - \pi \)

B. Area shared by \( r = 2 - 2\cos \theta \) and \( r = 2 \) \hspace{2cm} Answer: \( 5\pi - 8 \)

C. Inside \( r = 4\cos \theta \) and to the right of \( r = \sec \theta \) \hspace{2cm} Answer: \( \frac{8\pi}{3} + \sqrt{3} \)

D. Find the length of the curve over the given interval: \( r = \theta^2 \quad 0 \leq \theta \leq \sqrt{5} \) \hspace{2cm} Answer: \( \frac{19}{3} \)

E. Find the length of the curve over the given interval: \( r = \cos^3 \frac{\theta}{3} \quad 0 \leq \theta \leq \frac{\pi}{4} \) \hspace{2cm} Answer: \( \frac{\pi}{8} + \frac{3}{8} \)
Sequences

**Sequence:**

**Definition:** A sequence is a function whose domain \((n)\) is the set of positive integers. In other words, it is a list where order matters.

**Terms:** \(a_1, a_2, a_3, a_4, \ldots a_n\)

\(a_n\) is called the \(n^{th}\) term of the sequence and is denoted by \(\{a_n\}\)

Example 1: Write out the 1st four terms of the sequence: \(\{a_n\} = \left\{ \frac{2^n - 1}{2^n} \right\}\)

Find \(\lim_{n \to \infty} a_n = \)

If a sequence \(\{a_n\}\) agrees with a function \(f\) at every positive integer, and if \(f(x)\) approaches a limit \(L\) as \(x \to \infty\), then the sequence must **converge** to the same limit \(L\). If \(\lim_{x \to \infty} a_n\) does not exist, then \(a_n\) **diverges**.

**Theorem:**

Let \(L\) be a real number. Let \(f\) be a function of a real variable such that \(\lim_{x \to \infty} f(x) = L\).

If \(\{a_n\}\) is a sequence such that \(f(n) = a_n\) for every positive integer \(n\), then \(\lim_{n \to \infty} a_n = L\)

In other words: \(\lim_{n \to \infty} f(x) = \lim_{x \to \infty} a_n\) Why is this good?

Do the sequences converge or diverge? If the sequence converges, find its limit.

Ex. 2: \(a_n = (-1)^n \left( 1 + \frac{1}{n} \right)\)

Ex. 3: \(a_n = \frac{2n + 1}{1 + 3\sqrt{n}}\)
Ex. 4: \( a_n = \frac{n^2}{2^n - 1} \)

**Theorem:**

If \( \lim_{n \to \infty} |a_n| = 0 \), then \( \lim_{n \to \infty} a_n = 0 \)

Ex. 5: \( a_n = \left( -\frac{1}{2} \right)^n \) (look at graph)

Ex. 6: \( a_n = \frac{(-1)^n}{n} \) (look at graph)

**Monotonic Sequence:**

A sequence that is non-decreasing or non-increasing.

\( a_1 \leq a_2 \leq a_3 \leq a_4 \ldots \leq a_n \leq \ldots \) \quad or \quad \( a_1 \geq a_2 \geq a_3 \geq a_4 \ldots \geq a_n \geq \ldots \)

**Theorem:**

If a sequence \( \{a_n\} \) is bounded and monotonic, then it converges.

Look at the graph of: \( a_n = \frac{1}{n} \)
Look at the graph of: \( a_n = (-1)^n \)

Look at the graph of: \( a_n = \frac{n^2}{n+1} \) How could we prove that this is monotonic?

**Useful Limits to Know:**

\[
\lim_{n \to \infty} \frac{\ln n}{n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \sqrt{n} = 1
\]

\[
\lim_{n \to \infty} x^{1/n} = 1, \text{ for } x > 0 \quad \text{and} \quad \lim_{n \to \infty} x^n = 0, \text{ for } |x| < 1
\]

\[
\lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x \quad \text{and} \quad \lim_{n \to \infty} \frac{x^n}{n!} = 0
\]
Factorials:

\( n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n \) and for the special case of 0!, we define 0! = 1

Simplify the following expressions:

Ex. 6: \( \frac{(n-1)!}{(n+1)!} \)  
Ex. 7: \( \frac{(2n+1)!}{(2n)!} \)

Do the sequences converge or diverge? If the sequence converges, find its limit.

Ex. 8: \( a_n = n\pi \cos(n\pi) \)  
Ex. 9: \( a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right) \)

Ex. 10: \( a_n = \frac{(n-2)!}{n!} \)  
Ex. 11: \( a_n = \frac{5^n}{3^n} \)  
Ex. 12: \( a_n = \frac{e^n}{\pi^n} \)

11th ed. P. 596: 9, 17-37 odd AND show that \( a_n = \frac{n}{4n^2 + 1} \) is monotonic
**Series**

**Infinite Series:**

**Definition:** If \( \{a_n\} \) is an infinite sequence, then

\[
\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots + a_n + \cdots
\]

is an infinite series.

To find the sum of an infinite series, consider the **sequence** of partial sums. If the sequence of partial sums converges, then the series converges.

\[
S_1 = a_1 \\
S_2 = a_1 + a_2 \\
S_3 = a_1 + a_2 + a_3 \\
\ldots \\
S_n = a_1 + a_2 + a_3 + \cdots + a_n
\]

\[
\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k
\]

When this limit exists, we say that the series **converges**; otherwise, it **diverges**.

**Ex. 1:**

\[
\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots
\]

Sequence of Partial Sums:

**Ex. 2:**

\[
\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{n(n+1)} + \cdots
\]

Sequence of Partial Sums:
Conclusions:
For a convergent series, its **sum** is defined as the limit of its partial sums:

\[ S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k \]

In other words, if the sequence of partial sums \( \{S_n\} \) converges to \( S \),
then the series \( \sum_{n=1}^{\infty} a_n \) converges. The limit \( S \) is called the sum of the series.

If \( \{S_n\} \) diverges ( \( \lim_{n \to \infty} S_n \) does not exists), then the series \( \sum_{n=1}^{\infty} a_n \) diverges also.

**Problem:** \( S_n \) is often difficult if not impossible to calculate.

Special Series

**Geometric Series:**

\[ a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} = \sum_{n=1}^{\infty} ar^{n-1}, \quad a \neq 0 \]

where \( a \) is the first term and \( r \) is the common ratio.

\[ 5 + 15 + 45 + 135 + \cdots = \sum_{n=1}^{\infty} \]

\[ \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} \cdots = \sum_{n=1}^{\infty} \]

When will this series converge or diverge?

\[ r = 1 \quad r = -1 \]
What about other values of \( r \)?

\[
S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}
\]

\[
rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n
\]

**Conclusion:**

The series converges if __________________

Determine if the series converges or diverges. If it converges, find its sum.

Ex. 1: \( \sum_{n=1}^{\infty} 5 \cdot 3^{n-1} \) \hspace{1cm} Ex. 2: \( \sum_{n=1}^{\infty} 3 \left( \frac{2}{5} \right)^n \)
Ex. 3: \[ \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1}{3^n} \right) \]

Ex. 4: \[ \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{3^{2n}}{2^{3n+1}} \right) \]

Telescoping Series:

\[ (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots \]

Determine if the series converges or diverges. If it converges, find its sum.

Ex. 5: \[ \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) \]

Ex. 6: \[ \sum_{n=1}^{\infty} \ln \left( \frac{n+1}{n} \right) \]
Ex. 7: \( \sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} \)

**Nth Term Test**

\[ \sum_{n=1}^{\infty} a_n \to \text{the } a_n \text{'s must be getting smaller and smaller if the series is going to converge.} \]

However, if they are getting smaller it does NOT mean \( \sum_{n=1}^{\infty} a_n \) will converge.

Also, if the terms are not getting smaller it DOES mean it diverges.

- If \( \lim_{n \to \infty} a_n = 0 \), this tells us NOTHING \( a_n \)!
- If \( \lim_{n \to \infty} a_n \neq 0 \), this tells us the series diverges. The **Nth Term Test**.

Compare the sequence to the series:

\[ a_n = \frac{n+1}{2n+1} \quad \sum_{n=1}^{\infty} \frac{n+1}{2n+1} \]
Determine if the series converges or diverges. If it converges, find its sum (if possible).

Ex. 8: $\sum_{n=0}^{\infty} \left(\sqrt{2}\right)^n$

Ex. 9: $\sum_{n=1}^{\infty} (-1)^{n+1} n$

Ex. 10: $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{5^n}$

Ex. 11: $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$
**Integral Test and P-series**

**Integral Test:**
Let \( \{a_n\} \) be an infinite sequence of positive terms. If \( a_n = f(n) \), where \( f \) is continuous, positive, and decreasing on \([1, \infty)\), then

\[
\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_{1}^{\infty} f(x) \, dx
\]

either both converge or both diverge.

Use the integral test to see if the given series converges or diverges. (In order to use the integral test, you need to state that the function is continuous, positive, and decreasing).

Ex. 1: \( \sum_{n=1}^{\infty} \frac{1}{n} \)

Ex. 2: \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \)

**Note about sum:**
Power Series (P-Series)

These are series in the form: \[ \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^p = \sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots, \text{where } p > 0 \]

When \( p = 1 \), we get the harmonic series as seen in the first example:

\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots, \text{ and the related improper integral diverged:} \]

\[ \lim_{a \to \infty} \int_{1}^{a} \frac{1}{x} dx = \lim_{a \to \infty} (\ln(a) - \ln 1) = \infty \]

But what if \( p \neq 1? \)

Let \( f(x) = \frac{1}{x^p} \). This is a continuous, positive, decreasing function for \( x \geq 1 \)

\[ \lim_{a \to \infty} \int_{1}^{a} \frac{1}{x^p} dx = \lim_{a \to \infty} \int_{1}^{a} x^{-p} dx \]
Using the power rule to integrate, we get:

\[
\lim_{a \to \infty} \int_1^a x^{-p} \, dx = \lim_{a \to \infty} \left( \frac{x^{-p+1}}{-p+1} \right) \bigg|_1^a
\]

Now substituting in our values of \( a \) and 1:

\[
\lim_{a \to \infty} \left( \frac{x^{-p+1}}{-p+1} \right) \bigg|_1^a = \lim_{a \to \infty} \left( \frac{a^{-p+1} - 1^{-p+1}}{-p+1} \right) - \frac{1^{-p+1} - 1^{-p+1}}{-p+1}
\]

What happens when \( p < 1 \)? In other words, what happens when \( p \) is negative?

\[
\lim_{a \to \infty} \left( \frac{x^{-p+1}}{-p+1} \right) \bigg|_1^a = \lim_{a \to \infty} \left( \frac{a^{-p+1} - 1^{-p+1}}{-p+1} \right) = \infty
\]

And the integral diverges.

But what happens when \( p > 1 \)? In other words, what happens when \( p \) is positive?

\[
\lim_{a \to \infty} \left( \frac{x^{-p+1}}{-p+1} \right) \bigg|_1^a = \lim_{a \to \infty} \left( \frac{a^{-p+1} - 1^{-p+1}}{-p+1} \right) = \lim_{a \to \infty} \left( \frac{a^{1-p} - 1^{1-p}}{1-p} \right)
\]

\[
= \lim_{a \to \infty} \left( \frac{1}{a^{p-1} (1-p)} - \frac{1}{1-p} \right) = -\frac{1}{1-p}
\]

Since \((1-p)\) will now be a negative value, we can move it to the denominator.

And the integral converges.

Conclusions:

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1 \quad \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges if } p \leq 1
\]

We call this the P-series test.
Determine if the series converges or diverges.

Ex. 4: \[ \sum_{n=1}^{\infty} \frac{1}{2n - 1} \]

Ex. 5: \[ \sum_{n=1}^{\infty} \frac{3}{n^{5/3}} \]

Ex. 6: \[ \sum_{n=1}^{\infty} \frac{1}{n^{1/\pi}} \]

Ex. 7: \[ \sum_{n=1}^{\infty} \frac{n!}{n^5} \]

Comparisons of Series

Direct Comparison Test

Let \(0 < a_n \leq b_n\) for all \(n\).

1. If \(\sum_{n=1}^{\infty} b_n\) converges, then \(\sum_{n=1}^{\infty} a_n\) converges.

2. If \(\sum_{n=1}^{\infty} a_n\) diverges, then \(\sum_{n=1}^{\infty} b_n\) diverges.

Note: As stated, the Direct Comparison Test requires that \(0 < a_n \leq b_n\) for all \(n\). Because the convergence of a series is not dependent on its first several terms, you could modify the test to require only that \(0 < a_n \leq b_n\) for all \(n\) greater than some integer \(N\).

Limit Comparison Test

If \(a_n > 0, b_n > 0\) and \(\lim_{n \to \infty} \frac{a_n}{b_n} = L\)

1. Where \(L\) is finite and positive, then \(\sum_{n=1}^{\infty} a_n\) and \(\sum_{n=1}^{\infty} b_n\) either both converge or both diverge.

2. If the \(L = 0\) and \(\sum_{n=1}^{\infty} b_n\) converges, then \(\sum_{n=1}^{\infty} a_n\) converges.

3. If the \(L = \infty\) and \(\sum_{n=1}^{\infty} b_n\) diverges, then \(\sum_{n=1}^{\infty} a_n\) diverges.

The first case is the important one and it will work even if you accidentally write \(\frac{b_n}{a_n}\) instead of \(\frac{a_n}{b_n}\).

The second and third case require that you get the fraction "right side up".

Note: As with the Direct Comparison Test, the Limit comparison Test could be modified to require only that \(a_n\) and \(b_n\) be positive for all \(n\) greater than some integer \(N\).

Determine the convergence or divergence of the following series. Clearly state your reasons.

Ex. 1: \(\sum_{n=1}^{\infty} \frac{n-1}{n^4 + 2}\) by DCT and by the LCT
Ex. 2: $\sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}}$

Ex. 3: $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2 + 2}}$, by DCT

Ex. 4: $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2 + 2}}$, by LCT
Ex. 5: \[ \sum_{n=1}^{\infty} \frac{1 + \cos(n)}{n^2} \]

Ex. 6: \[ \sum_{n=1}^{\infty} \frac{5n^3 - 3n}{n^2(n-2)(n^2+5)} \]

Ex. 7: \[ \sum_{n=1}^{\infty} \frac{1}{2 + 3^n} \]

Ex. 8: \[ \sum_{n=1}^{\infty} \frac{\ln(n)}{n + 1} \]
Ex. 9: \( \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \)

Ex. 10: \( \sum_{n=1}^{\infty} \frac{4^n + 5}{7^n - 42} \)

11th ed. p. 620: 5-23 odd (use whatever test is best), 27-34 all
Alternating Series Test

Given a series: \[ \sum_{n=1}^{\infty} (-1)^n a_n \] or \[ \sum_{n=1}^{\infty} (-1)^{n+1} a_n \] where \( a_n > 0 \) (all terms are positive),

then the series converges if the following conditions are met:

1. \( \lim_{n \to \infty} a_n = 0 \) (If the limit \( \neq 0 \), then series will diverge by the nth-term test)

2. \( a_n \) eventually decreases. \( (a_{n+1} \leq a_n) \) To determine if this is true, you can take the derivative of the corresponding function. If the derivative is always negative, then the function is decreasing.

NOTE: This test does not tell anything about divergence.

1. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \)

2. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \)

3. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2 + 1} \)

4. \( \sum_{n=1}^{\infty} (-1)^n \ln \left(1 + \frac{1}{n}\right) \)
**Absolute Convergence Test**

If the series \( \sum_{n=1}^{\infty} |a_n| \) converges, then the series \( \sum_{n=1}^{\infty} a_n \) **converges absolutely.**

(The Ratio and Root Tests are tests for absolute convergence but not the ONLY tests.)

If \( \sum_{n=1}^{\infty} a_n \) converges BUT \( \sum_{n=1}^{\infty} |a_n| \) diverges, then \( \sum_{n=1}^{\infty} a_n \) **converges conditionally.**

Steps to determine absolute convergence:

1. Take the absolute value of the given series. Use any necessary test to determine convergence/divergence.
2. If absolute value of given series converges, then the original series converges ABSOLUTELY.
3. If absolute value of given series diverges, then you must go back to the original series and test it to determine convergence/divergence.
4. If the original series diverges, then the series diverges.
5. If the original series converges, then the series converges CONDITIONALLY.
Determine the convergence (absolute or conditional) or divergence of the following series. Clearly state your reasons.

Ex. 1: \[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n} \]
Ex. 2: \[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n} \]

Ex. 3: \[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(3+n)}{(5+n)} \]
Ex. 4: \[ \sum_{n=1}^{\infty} (-1)^{n} \frac{\sin(n)}{n^2} \]
Ex. 5: \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} \]
Ratio and Root Test

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms.

1. The series $\sum_{n=1}^{\infty} a_n$ converges absolutely when $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} < 1$.

2. The series $\sum_{n=1}^{\infty} a_n$ diverges when $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} > 1$ or $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \infty$.

3. The Ratio Test is inconclusive when $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = 1$

Note: Always use if you have factorials. Also, very useful if you have powers of $n$.

Root Test

1. The series $\sum_{n=1}^{\infty} a_n$ converges absolutely when $\lim_{n \to \infty} \sqrt[n]{|a_n|} < 1$.

2. The series $\sum_{n=1}^{\infty} a_n$ diverges when $\lim_{n \to \infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \infty$.

3. The Root Test is inconclusive when $\lim_{n \to \infty} \sqrt[n]{|a_n|} = 1$

Ex. 1: $\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

Ex. 2: $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$
Ex. 3: \( \sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n} \)

Ex. 4: \( \sum_{n=1}^{\infty} \left( 1 - \frac{3}{n} \right)^n \)

Review: \( \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x \)

Ex. 5. \( \sum_{n=1}^{\infty} \frac{n^n}{n!} \)

Ex. 6. \( \sum_{n=1}^{\infty} \frac{(n!)^2}{(3n)!} \)

Recall \( \lim_{n \to \infty} \sqrt[n]{n} = 1 \) 11th ed P. 637:17-35 odd (omit 33), 39-51 odd
Taylor and Maclaurin Series

Given \( f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots = \sum_{n=0}^{\infty} a_n(x-c)^n \) is differentiable (and therefore continuous) on the interval \((c-R,c+R)\) then the following is true:

\[
\begin{align*}
f^{(0)}(x) &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \cdots \\
f^{(1)}(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots \\
f^{(2)}(x) &= 2a_2 + 3!a_3(x-c) + 4\cdot 3a_4(x-c)^2 + \cdots \\
f^{(3)}(x) &= 3!a_3 + 4!a_4(x-c) + \cdots \\
& \vdots \\
f^{(n)}(x) &= n!a_n + (n+1)!a_{n+1}(x-c) + (n+2)!a_{n+2}(x-c)^2 + \cdots 
\end{align*}
\]

Evaluating each of these derivatives at \( x = c \) yields

\[
\begin{align*}
f^{(0)}(c) &= 0!a_0 \\
f^{(1)}(c) &= 1!a_1 \\
f^{(2)}(c) &= 2!a_2 \\
f^{(3)}(c) &= 3!a_3
\end{align*}
\]
and, in general, \( f^{(n)}(c) = n!a_n \). By solving for \( a_n \), you can find the coefficients of the power series representation of \( f(x) \) are: \( a_n = \frac{f^{(n)}(c)}{n!} \).

\[
f(c) = a_0, \quad f'(c) = a_1, \quad \frac{f''(c)}{2!} = a_2, \ldots, \quad a_n = \frac{f^{(n)}(c)}{n!}
\]

Definitions of nth Taylor Polynomial and nth Maclaurin Polynomial

If \( f \) has \( n \) derivatives at \( c \), then the polynomial

\[
P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n
\]

is called the \( \text{nth Taylor polynomial for } f \text{ at } c \). If \( c = 0 \), then

\[
P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n
\]

is called the \( \text{nth Maclaurin polynomial for } f \).
Derivation of series for $e^x$, $\cos x$, $\sin x$, and $\ln x$

$f(x) = e^x$  

$f(x) = \cos(x)$ and $f(x) = \sin(x)$

$f(x) = \ln(x)$
Maclaurin Series for a Composite Function:

Find the Maclaurin series.
1. \( f(x) = \sin x^2 \)
2. \( f(x) = \cos \sqrt{x} \)
3. \( f(x) = e^{-3x} \)
4. \( f(x) = \ln(1 + x^2) \)

Find nth degree Taylor or Maclaurin series

1. \( f(x) = e^{-x}, \quad n = 5, \quad c = 0 \)
2. \( f(x) = \cos \pi x, \quad n = 4, \quad c = 0 \)
3. \( f(x) = \sqrt[3]{x}, \quad n = 3, \quad c = 8 \)
**Power Series**

**Definition**

If \( x \) is a variable, then an infinite series of the form

\[
\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots
\]

is called a **power series**. More generally, an infinite series of the form

\[
\sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1 (x - c) + a_2 (x - c)^2 + \cdots + a_n (x - c)^n + \cdots
\]

is called a **power series centered at** \( c \), where \( c \) is a constant.

A power series in \( x \) can be viewed as a function of \( x \)

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n
\]

where \( (x - c)^0 \) is defined to \( = 1 \) when \( x = c \)

The **domain of** \( f \) is the set of all \( x \) for which the power series converges. Determination of the domain of a power series is the primary concern of this topic. Of course, every power series converges at its center \( c \) because

\[
f(c) = a_0 + a_1 (c - c) + a_2 (c - c)^2 + \cdots + a_n (c - c)^n + \cdots
\]

\[
= a_0 + 0 + 0 + 0 + \cdots
\]

\[
= a_0
\]

Thus \( c \), always lies in the domain of \( f \). The domain of a power series can take three basic forms:

1. **A single point**
   
   \[
   c
   \]
   
   \[
   x
   \]

2. **An interval**
   
   \[
   c
   \]
   
   \[
   R
   \]
   
   \[
   R
   \]
   
   \[
   x
   \]

3. **The real number line**
   
   \[
   c
   \]
   
   \[
   x
   \]

The convergence of the series will depend upon the value of \( x \) that we put into the series.

**Theorem:**

For a power series at \( x = c \) and \( R = \) the **radius of convergence**, exactly one of the following is true:

1. The series converges at \( x = c \). The domain is only \( x = c \). \( R = 0 \), and the interval of convergence is \([c]\).
2. The series converges absolutely for \( |x - c| < R \) and diverges for \( |x - c| > R \)

   If \( |x - c| < R \), then the interval of convergence can be

   \[
   (c - R, c + R) \text{ or } [c - R, c + R] \text{ or } [c - R, c + R]
   \]

   You must check the endpoints.

3. The series converges for all \( x \). If \( R = \infty \), the interval of convergence is \((\infty, \infty)\)
Find the interval of convergence for each series.

Ex. 1: \( \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} \)

Ex. 2: \( \sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n n!} \)
Ex. 3: \( \sum_{n=1}^{\infty} n!(x - 4)^n \)

Ex. 4: \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{3 + \sqrt{n}} \)
Ex. 5: \[
\sum_{n=1}^{\infty} \frac{(3x - 2)^n}{n}
\]
Find the interval of convergence for each series.

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<td>1.</td>
<td>[ \sum_{n=0}^{\infty} x^n ]</td>
<td>6.</td>
<td>[ \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} ]</td>
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<td>2.</td>
<td>[ \sum_{n=0}^{\infty} (-1)^n (4x + 1)^n ]</td>
<td>7.</td>
<td>[ \sum_{n=1}^{\infty} \frac{4^n x^{2n}}{n} ]</td>
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<td>3.</td>
<td>[ \sum_{n=0}^{\infty} \frac{(x - 2)^n}{10^n} ]</td>
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<td>[ \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2 + 3}} ]</td>
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<td>4.</td>
<td>[ \sum_{n=0}^{\infty} \frac{nx^n}{(n + 2)} ]</td>
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<td>[ \sum_{n=0}^{\infty} \frac{n(x + 3)^n}{5^n} ]</td>
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<td>5.</td>
<td>[ \sum_{n=1}^{\infty} \frac{x^n}{n \sqrt{n^3}} ]</td>
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Answers:

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<td>1.</td>
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<td>2.</td>
<td>(-\frac{1}{2}, 0)</td>
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<td>(\left(\frac{1}{2}, \frac{1}{2}\right))</td>
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<td>3.</td>
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Representation of Functions as Power Series

Geometric Power Series

Consider the function \( f(x) = \frac{1}{1-x} \). The form of this function closely resembles the sum of a geometric series: \( \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, 0 < |r| < 1 \). In other words, when \( a = 1 \) and \( r = x \), a power series representation for \( \frac{1}{1-x} \) centered at 0 is

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots, |x| < 1
\]

Finding a Geometric Power Series centered at 0

Ex. 1: \( f(x) = \frac{1}{1 + x^2} \)

\( f(x) = \frac{1}{1-x} \), Domain: all \( x \neq 1 \)

Ex. 2: \( f(x) = \frac{5}{2x + 3} \)

\( f(x) = \sum_{n=0}^{\infty} x^n \), Domain: \( -1 < x < 1 \)
Ex. 3: \( f(x) = \frac{x^3}{x+2} \)

Ex. 4: \( f(x) = \frac{3}{x^2 - x - 2} \)
Differentiation and Integration of Power Series

We can do this by differentiating or integrating each individual term in the series, just as we would for a polynomial. This is called **term-by-term differentiation and integration**.

If the power series \( \sum_{n=0}^{\infty} a_n (x - c)^n \) has a radius of convergence \( R > 0 \), then the function defined by \( f(x) = a_0 + a_1 (x - c) + a_2 (x - c)^2 + \ldots = \sum_{n=0}^{\infty} a_n (x - c)^n \) is differentiable (and therefore continuous) on the interval \((c - R, c + R)\) and the following is true:

1. \( f'(x) = a_1 + 2a_2 (x - c) + 3a_3 (x - c)^2 + \ldots = \sum_{n=1}^{\infty} a_n n(x - c)^{n-1} \)

2. \( \int f(x) \, dx = C + a_0 (x - c) + a_1 \frac{(x - c)^2}{2} + a_2 \frac{(x - c)^3}{3} + \ldots = C + \sum_{n=0}^{\infty} a_n \frac{(x - c)^{n+1}}{n + 1} \)

Express the following as a power series:

Ex 1: \( f(x) = \ln(1 - x) \)  
Ex 2: \( f(x) = \arctan x \)
Ex 3: \( f(x) = \frac{1}{(1-x)^2} \)

What if we multiplied problems 1-3 by \( x \)?

Ex 4: \( \int_{0}^{1} \frac{1}{1 + x^7} \, dx \)
Power Series Representation

Find a power series representation for the function:

1. \( f(x) = \frac{1}{1+x} \)
2. \( f(x) = \frac{2}{3-x} \)
3. \( f(x) = \frac{3}{2x-1} \)
4. \( f(x) = \frac{1}{2x-5} \)
5. \( f(x) = \frac{4}{3x+2} \)
6. \( f(x) = \frac{3x}{x^2+x-2} \)
7. \( f(x) = \frac{4x-7}{2x^2+3x-2} \)
8. \( f(x) = \frac{x}{9+x^2} \)
9. \( f(x) = \frac{x}{2x^2+1} \)
10. \( f(x) = \frac{4}{4+x^2} \)
11. \( f(x) = \frac{3x^2}{9-x} \)
12. \( f(x) = \ln(x+4) \)
13. \( f(x) = x \arctan(2x) \)
14. \( f(x) = \frac{1}{(x-3)^2} \)
15. \( f(x) = \frac{1}{(1-2x)^2} \)
16. \( \int_0^1 \frac{1}{1+x^5} \, dx \)
1. \( f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \)

2. \( f(x) = \frac{2}{3-x} = 2 \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}} \)

3. \( f(x) = \frac{3}{2x-1} = -3 \sum_{n=0}^{\infty} (2x)^n \)

4. \( f(x) = \frac{1}{2x-5} = -\sum_{n=0}^{\infty} (2x)^n \)

5. \( f(x) = \frac{4}{3x+2} = \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^n}{2^{n-1}} \)

6. \( f(x) = \frac{3x}{x^2+x-2} = \frac{2}{x+2} + \frac{1}{x-1} \)
   \( = \sum_{n=0}^{\infty} (-1)^n \left( \frac{x^n}{2} \right) - \sum_{n=0}^{\infty} x^n \)

7. \( f(x) = \frac{4x-7}{2x^2+3x-2} = \frac{-2}{(2x-1)^2} + \frac{3}{x+2} \)
   \( = \sum_{n=0}^{\infty} (2)^{n+1} x^n + 3 \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}} \)

8. \( f(x) = \frac{x}{9+x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{9^{n+1}} \)

9. \( f(x) = \frac{x}{2x^2+1} = \sum_{n=0}^{\infty} (-2)^n x^{2n+1} \)

10. \( f(x) = \frac{4}{4+x^2} = \sum_{n=0}^{\infty} \left( \frac{1}{4} \right)^n x^{2n} \)

11. \( f(x) = \frac{3x^2}{9-x} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{3^{2n+1}} \)

12. \( f(x) = \ln(x+4) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} (n+1) + \ln 4 \)

13. \( f(x) = x \arctan(2x) = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} \frac{x^{2n+2}}{2n+1} \)

14. \( f(x) = \frac{1}{(x-3)^2} = \sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n nx^{n-1} \)
   \( OR \quad \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^{n+2} (n+1)x^n \)

15. \( f(x) = \frac{1}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^{n-1} nx^{n-1} \)
   \( OR \quad \sum_{n=0}^{\infty} 2^n (n+1)x^n \)

16. \( \int_0^1 \frac{1}{1+x^2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1} \)