Math 1350 Project Problems for Chapters 3, 4, and 5(Due by EOC Nov. 20)

Mind Your Four’s And Two’s

1. What is the value of \( x \) if \( 8^{20,000} + 8^{20,000} = 2^x \)?
   [Hint: Factor \( 8^{20,000} + 8^{20,000} \) and use the fact that \( 8 = 2^3 \).]

A Lot Of Weeks, But How Many Days Left Over?

2. What is the remainder when \( 2^{15,110} \) is divided by 7?
   [Hint: Look for a pattern in the remainders:
   
   \[
   \begin{array}{c|c}
   \text{Power of 2} & \text{Remainder when divided by 7} \\
   \hline
   2^1 = 2 & 2 \\
   2^2 = 4 & 4 \\
   2^3 = 8 & 1 \\
   2^4 = 16 & 2 \\
   2^5 = 32 & 4 \\
   \vdots & \vdots \\
   \end{array}
   \]
   
   .]

Seven Heaven or Seven…

3. Find the largest power of 7 that divides 343!.
   [Hint: The multiples of 7 occurring in the expansion of 343! are 7,14,21,28,\ldots,7\cdot 49.
   
   The multiples of \( 7^2 = 49 \) occurring in the expansion of 343! are 49,98,\ldots,49\cdot 7
   
   The multiple of \( 7^3 = 343 \) occurring in the expansion of 343! is just 343.
   
   There are no multiples of higher powers of 7 occurring in the expansion of 343!]

Who Needs Logarithms?

4. If \( 2^x = 15 \) and \( 15^y = 32 \), then find the value of \( xy \).
   [Hint: Substitute the first equation into the second equation, and use an exponent property.]
Can You Just Tell Me How Old Your Children Are?

5. A student asked his math teacher, “How many children do you have, and how old are they?” “I have 3 girls,” replied the teacher. “The product of their ages is 72, and the sum of their ages is the same as the room number of this classroom.” Knowing that number, the student did some calculations and said, “There are two solutions.” “Yes, that is so,” said the teacher, “but I still hope that the oldest will some day win a math prize at this school.” The student then gave the ages of the three girls. What are the ages?

\[\text{Hint:}\]

<table>
<thead>
<tr>
<th>Triple factors of 72</th>
<th>Sum of the factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,1,72</td>
<td>74</td>
</tr>
<tr>
<td>1,2,36</td>
<td>39</td>
</tr>
<tr>
<td>1,3,24</td>
<td>28</td>
</tr>
<tr>
<td>1,4,18</td>
<td>23</td>
</tr>
<tr>
<td>1,6,12</td>
<td>19</td>
</tr>
<tr>
<td>1,8,9</td>
<td>18</td>
</tr>
<tr>
<td>2,2,18</td>
<td>22</td>
</tr>
<tr>
<td>2,3,12</td>
<td>17</td>
</tr>
<tr>
<td>2,4,9</td>
<td>15</td>
</tr>
<tr>
<td>2,6,6</td>
<td>14</td>
</tr>
<tr>
<td>3,3,8</td>
<td>14</td>
</tr>
<tr>
<td>3,4,6</td>
<td>13</td>
</tr>
</tbody>
</table>

Cover All Your Bases, If It’s Within Your Power.

6. Solve for \(x\) if \(\left(x^2 - 5x + 5\right)^{x^2 - 9x + 20} = 1\).

\[\text{Hint: Any number raised to the zero power, except zero itself, equals 1. } 1 \text{ raised to any power is equal to 1. } -1 \text{ raised to an even power is equal to 1.}\]

A Special Case Of The Chinese Remainder Theorem.

7. The positive integer \(n\), when divided by 3, 4, 5, 6, and 7, leaves remainders of 2, 3, 4, 5, and 6, respectively. Find the smallest possible value of \(n\).

\[\text{Hint: } n = 3a + 2, n = 4b + 3, n = 5c + 4, n = 6d + 5, n = 7e + 6.\]

This means that
\[n+1 = 3(a+1), n+1 = 4(b+1), n+1 = 5(c+1), n+1 = 6(d+1), n+1 = 7(e+1).\]

So \(n+1\) is a common multiple of 3, 4, 5, 6, and 7. What’s the least common multiple?\}
8. How many zeroes are at the end of the number 127!?  
\[ 127! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots 126 \cdot 127 \]

\{Hint: Zeroes come from factors of 10. Factors of 10 come from 5’s and 2’s. See the hint for problem #3.\}

9. How many zeroes are at the end of the number \( 2^{3000} \cdot 5^{6000} \cdot 4^{4000} \)?

\{Hint: Zeroes come from factors of 10. Factors of 10 come from 5’s and 2’s.\}

10. Foghorn C sounds every 34 seconds, and foghorn D sounds every 38 seconds. If they sound together at noon, what time will it be when they next sound together?

<table>
<thead>
<tr>
<th>Foghorn C</th>
<th>12:00</th>
<th>12:00:34</th>
<th>12:01:08</th>
<th>12:01:42</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>Foghorn D</td>
<td>12:00</td>
<td>12:00:38</td>
<td>12:01:16</td>
<td>…</td>
<td>…</td>
</tr>
</tbody>
</table>

\{Hint: Every time they sound together after noon will have to be both a multiple of 34 seconds after noon and a multiple of 38 seconds after noon.\}

11. How many digits does the number \( 8^{6,666} \cdot 5^{20,000} \) have?

\{Hint: Zeroes come from factors of 10. Factors of 10 come from 5’s and 2’s.\}
Don’t Get Stumped; Use The Fundamental Theorem Of Arithmetic.

12. Forrest Stump heard that there are only two numbers between 2 and 300,000,000,000,000 which are perfect squares, perfect cubes, and perfect fifth powers. He decided to look for them, and so far he has checked out every number up to about 100,000 and is beginning to get discouraged. What are the numbers he is trying to find?

   \[ \text{Hint: Every positive whole number greater than 1 can be written as a product of prime factors. If } N \text{ is a positive whole number greater than 2, then } N = 2^n \cdot 3^m \cdot 5^k \cdot \ldots \cdot p_k^{n_k}. \text{ In order for } N \text{ to be a perfect square, all the positive exponents would have to be multiples of 2; in order for } N \text{ to be a perfect cube, all the positive exponents would have to be multiples of 3; and in order for } N \text{ to be a perfect fifth power, all the positive exponents would have to be multiples of 5. So all the positive exponents would have to be common multiples of 2, 3, and 5.} \]

Round And Round With Donald And Hillary

13. Donald and Hillary are racing cars around a track. Donald can make a complete circuit in 72 seconds, and Hillary completes a circuit in 68 seconds.

   a) If they start together at the starting line, how many seconds will it take for Hillary to pass Donald at the starting line for the first time.

   \[ \text{Hint: Every time Donald reaches the starting line must be a multiple of 72 seconds, and every time Hillary reaches the starting line must be a multiple of 68 seconds. So they will both be at the starting line at common multiples of 72 and 68.} \]

   b) If they start together, how many laps will Donald have completed when Hillary has completed one more lap than Donald?

   \[ \text{Hint: Let } n \text{ be the number of laps completed by Donald, then } 72n = 68(n+1). \]
Some divisibility rules for positive integers:

1) A positive integer is divisible by 2 if its one’s digit is even.
   
   Here’s why:
   Suppose we have the three-digit integer $abc$, then its value can be expressed as $100a + 10b + c$, but $100a + 10b + c = 2(50a + 5b) + c$, so if the one’s digit, $c$ is even (divisible by 2) then the integer $abc$ will also be divisible by 2.

2) A positive integer is divisible by 3 if the sum of its digits is divisible by 3. This process may be repeated.
   Examples: 243 is divisible by 3, but 271 is not.
   
   Here’s why:
   Suppose we have the three-digit integer $abc$, then its value can be expressed as $100a + 10b + c$, but $100a + 10b + c = 99a + 9b + a + b + c = 3(33a + 3b) + (a + b + c)$, so if the sum of the digits, $a + b + c$, is divisible by 3 then the integer $abc$ will also be divisible by 3.

3) A positive integer is divisible by 4 if the ten’s and one’s digits form a two-digit integer divisible by 4.
   Examples: 724 is divisible by 4, but 726 is not.
   
   Here’s why:
   Suppose we have the four-digit integer $abcd$, then its value can be expressed as $1000a + 100b + 10c + d$, but $1000a + 100b + 10c + d = (1000a + 100b) + (10c + d) = 4(250a + 25b) + (10c + d)$, so if the ten’s and one’s two-digit integer, $cd$, is divisible by 4 then the integer $abcd$ will also be divisible by 4.

4) A positive integer is divisible by 5 if its one’s digit is either a 5 or a 0.
   
   Here’s why:
   Suppose we have the three-digit integer $abc$, then its value can be expressed as $100a + 10b + c$, but $100a + 10b + c = 5[20a + 2b] + c$, so if the one’s digit, $c$ is divisible by 5, then the integer $abc$ will also be divisible by 5, but the only digits divisible by 5 are 0 and 5.

5) A positive integer is divisible by 6 if it’s both divisible by 2 and divisible by 3.
6) A positive integer is divisible by 7 if when you remove the one’s digit from the integer and then subtract twice the one’s digit from the new integer, you get an integer divisible by 7. This process may be repeated.
Examples: 714 is divisible by 7 since 71 – 8 = 63, but 423 is not since 42 – 6 = 36.

Here’s why:
Suppose we have the three-digit integer $abc$, then its value can be expressed as $100a + 10b + c$, but
$$
100a + 10b + c = 90a + 9b + 3c + (10a + b - 2c)
$$
new integer minus twice one's digit
$$
= 9(10a + b - 2c) + 21c + (10a + b - 2c)
$$
new integer minus twice one's digit
$$
= 10 \cdot (10a + b - 2c) + 7 \cdot (3c)
$$
new integer minus twice one's digit
, so if the new integer minus twice the one’s digit, $10a + b - 2c$, is divisible by 7 then so is the original integer $abc$.

Or

A positive integer is divisible by 7 if when you remove the one’s digit from the integer and then subtract nine times the one’s digit from the new integer, you get an integer divisible by 7. This process may be repeated.
Examples: 714 is divisible by 7 since 71 – 36 = 35, but 423 is not since 42 – 27 = 15.

Here’s why:
Suppose we have the three-digit integer $abc$, then its value can be expressed as $100a + 10b + c$, but
$$
100a + 10b + c = 90a + 9b + 10c + (10a + b - 9c)
$$
new integer minus 9 times one's digit
$$
= 9(10a + b - 9c) + 91c + (10a + b - 9c)
$$
new integer minus 9 times one's digit
$$
= 10 \cdot (10a + b - 9c) + 7 \cdot (13c)
$$
new integer minus 9 times one's digit
, so if the new integer minus nine times the one’s digit, $10a + b - 9c$, is divisible by 7 then so is the original integer $abc$.

Or

A positive integer with more than three digits is divisible by 7 if when you split the digits into groups of three starting from the right and alternately add and subtract these three digit numbers you get a result which is divisible by 7.
Examples: 1412236 is divisible by 7 since $1 - 412 + 236 = -175$, but 130747591 is not since $130 - 747 + 591 = -26$.
Here’s why:
Suppose we have the five-digit integer \(abcde\), then its value can be expressed as
\[
10000a + 1000b + 100c + 10d + e \quad \text{but}
\]
\[
10000a + 1000b + 100c + 10d + e = (10001a + 1001b) + (10a + b) - (100c + 10d + e)
\]
\[
+ 2(100c + 10d + e) - 2(10a + b)
\]
\[
= 7 \cdot 143(10a + b) - \left( \frac{10a + b}{\text{integer ab}} - \frac{100c + 10d + e}{\text{integer cde}} \right)
\]
integer \(ab\) minus the integer \(cde\) is divisible by 7 then the integer \(abcde\) will also be divisible by 7.

\textbf{7) A positive integer is divisible by 8 if the hundred’s, ten’s, and one’s digits form a three-digit integer divisible by 8.}
Examples: 1240 is divisible by 8, since 240 is, 3238 is not, since 238 is not even divisible by 4.

Here’s why:
Suppose we have the five-digit integer \(abcde\), then its value can be expressed as
\[
10000a + 1000b + 100c + 10d + e \quad \text{but}
\]
\[
10000a + 1000b + 100c + 10d + e = (10001a + 1001b) + (100c + 10d + e)
\]
\[
= 8(1250a + 125b) + \left( \frac{100c + 10d + e}{\text{the three-digit integer cde}} \right) \quad \text{so if the hundred’s, ten’s, and one’s three-digit integer, \(cde\), is divisible by 8 then the integer \(abcde\) will also be divisible by 8.}
\]

\textbf{8) A positive integer is divisible by 9 if the sum of its digits is divisible by 9. This process may be repeated.}
Examples: 243 is divisible by 9, but 9996 is not.

Here’s why:
Suppose we have the three-digit integer \(abc\), then its value can be expressed as \(100a + 10b + c\) , but \(100a + 10b + c = 99a + 9b + a + b + c = 9(11a + b) + (a + b + c)\) , so if the sum of the digits, \(a + b + c\), is divisible by 9 then the integer \(abc\) will also be divisible by 9.

\textbf{9) A positive integer is divisible by 11 if when you remove the one’s digit from the integer and then subtract the one’s digit from the new integer, you get an integer divisible by 11. This process may be repeated.}
Examples: 1001 is divisible by 11 since \(100 - 1 = 99\), but 423 is not since \(42 - 3 = 38\).
Here’s why:
Suppose we have the three-digit integer $abc$, then its value can be expressed as $100a + 10b + c$, but

\[
100a + 10b + c = 90a + 9b + 2c + (10a + b - c) \\
= 9(10a + b - c) + 11c + (10a + b - c) \\
= 10 \cdot (10a + b - c) + 11c
\]

, so if the new integer minus the one’s digit, $10a + b - c$, is divisible by 11 then so is the original integer $abc$.

Or

A positive integer is divisible by 11 if when you subtract the sum of the ten’s digit and every other digit to the left from the sum of the one’s digit and every other digit to the left you get a number divisible by 11. This process may be repeated.
Examples: 9031 is divisible by 11 since $(1+0)-(3+9)=-11$, but 423 is not since $(4+3)-(2)=5$.

Here’s why:
Suppose we have the five-digit integer $abcde$, then its value can be expressed as $10,000a + 1,000b + 100c + 10d + e$, but

\[
10,000a + 1,000b + 100c + 10d + e = (9,999+1)a + (1,001-1)b + (99+1)c + (11-1)d + e \\
= 9,999a + 1,001b + 99c + 11d + (a + c + e) - (b + d) \\
= 11 \cdot (909a + 91b + 9c + d) + \left[ (a + c + e) - (b + d) \right]
\]

, so if the sum of the alternating digits from the one’s digit minus the sum of the alternating
digits from the ten’s digit, $(a + c + e) - (b + d)$, is divisible by 11 then so is the original integer $abcde$.

Or

A positive integer with more than three digits is divisible by 11 if when you split the digits into groups of three starting from the right and alternately add and subtract these three digit numbers you get a result which is divisible by 11.
Examples: 1412290 is divisible by 7 since $1 - 412 + 290 = -121$, but 130747591 is not since $130 - 747 + 591 = -26$.

Here’s why:
Suppose we have the five-digit integer $abcde$, then its value can be expressed as $10000a + 1000b + 100c + 10d + e$, but
\[10000a + 1000b + 100c + 10d + e = (10010a + 1001b) + (10a + b) - (100c + 10d + e)\]
\[+2(100c + 10d + e) - 2(10a + b)\]
\[= 11 \cdot 91(10a + b) - \left[\frac{10a + b}{\text{integer } ab} - \frac{100c + 10d + e}{\text{integer } cde}\right]\]

so if the integer \(ab\) minus the integer \(cde\) is divisible by 11 then the integer \(abcde\) will also be divisible by 11.

### Divisibility And Conquer.

14. Of the following three numbers determine which are divisible by 2, which are divisible by 3, which are divisible by 4, which are divisible by 6, which are divisible by 8, and which are divisible by 9: \(3 \times 10^{999} + 736, 3 \times 10^{999} + 534, 3 \times 10^{999} + 952\).

\{Hint: Divisibility rules.\}

### Keep On Dividing And Conquering.

15. Of the following three numbers determine which are divisible by 3, which are divisible by 6, which are divisible by 7, which are divisible by 9, and which are divisible by 11: \(1358024680358024679, 864197523864197523, 964197523864197522\).

\{Hint: Divisibility rules.\}

### Seven Come Thirteen, Not Eleven.

16. Show that the second divisibility rule for 7 can also be used as a divisibility rule for 13. Modify the explanation to show why it works for 13.

### A Whole Lotta Eights; A Whole Lotta…

17. a) What is the smallest whole number that when multiplied by 9 gives a number whose digits are all 8’s?

b) What is the smallest whole number that when multiplied by 9 gives a number whose digits are all 5’s?

c) What is the smallest whole number that when multiplied by 9 gives a number whose digits are all 3’s? \{Be careful!\}
### Which Digits?

18. **a)** Given that the number 1234567891234567895913d8 is divisible by 12, what are the possible values for the digit d?

   *It must be divisible by 3, so what can you say about the digit sum? It must be divisible by 4, so what can you say about the two-digit number d8?*

   **b)** Given that the number 1134567891234567895913d8 is divisible by 24, what are the possible values for the digit d?

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### Crazy 8.

19. Show that the difference of the squares of two odd numbers is divisible by 8.

   *Hint: Suppose the two odd numbers are \( x = 2n + 1 \) and \( y = 2m + 1 \).
   
   Then \( x^2 - y^2 = (2n + 1)^2 - (2m + 1)^2 = (4n^2 + 4n + 1) - (4m^2 + 4m + 1) = 4(n - m)(n + m + 1) \).

   *If you can show that \( (n - m)(n + m + 1) \) must be divisible by 2, then you’re done.*

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m )</th>
<th>( n - m )</th>
<th>( n + m + 1 )</th>
<th>( (n - m)(n + m + 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>even</td>
<td>even</td>
<td>odd</td>
<td>?</td>
</tr>
<tr>
<td>odd</td>
<td>odd</td>
<td>even</td>
<td>odd</td>
<td>?</td>
</tr>
<tr>
<td>even</td>
<td>odd</td>
<td>odd</td>
<td>even</td>
<td>?</td>
</tr>
<tr>
<td>odd</td>
<td>even</td>
<td>odd</td>
<td>even</td>
<td>?</td>
</tr>
</tbody>
</table>

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### That’s Sum Divisibility Rule!

20. If \( x \) has an even number of digits and \( y \) has the same digits but in opposite order, then what number must the sum of \( x \) and \( y \) be divisible by?

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### Lucky 13, I Repeat, Lucky 13. Lucky 7, I Repeat….

21. **a)** Show that every six-digit number of the form \( abc,abc \) (for example 281,281 or 435,435) is divisible by 13.

   **b)** Show that every six-digit number of the form \( abc,abc \) (for example 281,281 or 435,435) is divisible by 7.

   **c)** Show that every six-digit number of the form \( abc,abc \) (for example 281,281 or 435,435) is divisible by 11.
It’s Just Gotta Be 6.

22. a) Show that for any integer \( n \), 6 divides \( n^3 - n \).

\[ \text{Hint: } n^3 - n = n(n^2 - 1) = (n-1) \cdot n \cdot (n+1) \]

b) Show that for any integer \( n \), 6 divides \( n^5 - n \).

\[ \text{Hint: } n^5 - n = n(n^4 - 1) = (n-1) \cdot n \cdot (n+1) \cdot (n^2 + 1) \]

Cogswell Cogs Or Spacely Sprockets?

23. In a machine, a small gear with 45 teeth is engaged with a large gear with 96 teeth. How many more revolutions will the smaller gear have made than the larger gear the first time the two gears are in their starting position?

\[ \text{Hint: A revolution of the smaller gear is a multiple of 45 teeth, and a revolution of the larger gear is a multiple of 96 teeth. So the gears are again in the starting positions at common multiples of 45 and 96.} \]

My Tile Cutter Is Broken, So What Now?

24. The figure shows that twenty-four 8"\(\times\)12" rectangular tiles can be used to tile a 48"\(\times\)48" square without cutting tiles.

a) Is there a smaller sized square that can be tiled without cutting using 8"\(\times\)12" tiles? If so, find it.

\[ \text{Hint: The dimension of the square must be a multiple of both dimensions of the rectangular tile.} \]

b) What is the smallest square that can be tiled without cutting using 9"\(\times\)12" tiles?

\[ \text{Hint: See the previous hint.} \]
The Great Luggage Caper.
25. Great Aunt Christine is going for her annual holiday to Barbados. She sends her butler John down to the airport with her collection of suitcases, each of which weighs either 18 or 84 pounds, and is informed that the total weight checked-in is 652 pounds. Show that this is impossible without listing and checking all the possible combinations of 18 and 84 pound suitcases.

{Hint: The total weight of the suitcases is of the form 18x + 84y, where x and y are nonnegative integers. So the expression must be divisible by the greatest common factor of 18 and 84.}

I Hate This Problem To The Nth Degree.
26. Use the following properties of exponents to find the exact value of the given expressions.

\[ x^n \cdot x^m = x^{n+m}, \quad (xy)^n = x^n y^n, \quad (x^n)^m = x^{nm} \]

a) \[ \frac{3^{90,000} \cdot 6^{90,000}}{2^{89,999} \cdot 9^{90,000}} \]

b) \[ \frac{2^{100,000} \cdot 3^{100,000}}{6^{100,000} \cdot 9^{5000}} \]

Logarithms Are Very Overrated.
27. If \( 3^a = 4 \), \( 4^b = 5 \), \( 5^c = 6 \), \( 6^d = 7 \), \( 7^e = 8 \), and \( 8^f = 9 \), then what is the value of the product \( abcdef \)?

{Hint: From \( 3^a = 4 \) and \( 4^b = 5 \), you can conclude that \( 3^{ab} = 5 \). So just keep going.}

Given The Least, What’s The Greatest?
28. a) Given that \( 12^{10} \) is the least common multiple of the positive integers \( 6^6, 6^{10}, \) and \( k \). How many different values of \( k \) are possible?

b) What is the least number of prime numbers (not necessarily different) that 3,185 must be multiplied by so that the product is a perfect cube?
Don’t Put All Of Your Eggs In One Basket!

29. If eggs are taken from a basket two at a time, then one egg remains in the basket. If eggs are taken three at a time from the same basket, then two eggs remain in the basket. If eggs are taken four at a time from the same basket, then three eggs remain in the basket. If eggs are taken five at a time from the same basket, then four eggs remain in the basket. If eggs are taken six at a time from the same basket, then five eggs remain in the basket. If eggs are taken seven at a time from the same basket, then no eggs remain in the basket. What is the fewest possible number of eggs in the basket?

[Hint: If \( N \) is the number of eggs in the basket, then \( N + 1 \) must be a multiple of 2, \( N + 1 \) must be a multiple of 3, \( N + 1 \) must be a multiple of 4, \( N + 1 \) must be a multiple of 5, and \( N + 1 \) must be a multiple of 6. So \( N + 1 \) must be a multiple of the LCM of 2, 3, 4, 5, and 6. Also, \( N \) must be a multiple of 7.]

Given The Greatest, What’s The Least?

30. If the product of two natural numbers is \( 2^73^85^{11} \), and their greatest common factor is \( 2^33^45^1 \), then what is their least common multiple?

The Euclidean algorithm can be used to find the greatest common factor of two whole numbers. It can also be used to express the greatest common factor as a combination of the two whole numbers. Here’s an example:

Let’s find the gcf of 738 and 621 using the Euclidean algorithm, and express it as a combination of 738 and 621.

\[
\begin{align*}
738 &= 1 \cdot 621 + 117 \\
621 &= 5 \cdot 117 + 36 \\
117 &= 3 \cdot 36 + 9 \\
36 &= 4 \cdot 9
\end{align*}
\]

So the gcf of 738 and 621 is 9. Now work backwards through the equations.

\[
\begin{align*}
9 &= 117 - 3 \cdot 36 = 117 - 3(621 - 5 \cdot 117) = 16 \cdot 117 - 3 \cdot 621 = 16(738 - 621) - 3 \cdot 621 \\
&= 16 \cdot 738 - 19 \cdot 621.
\end{align*}
\]

So \( 16 \cdot 738 - 19 \cdot 621 = 9 \).

The Greatest Common Combination.

31. Find the greatest common factor of the following pairs of numbers, and express it as a combination of the two numbers.

a) 24 and 54  

b) 72 and 160  

c) 5291 and 11951
The greatest common factor of 7 and 9 is 1. The equation $7 \cdot (-5) + 9 \cdot 4 = 1$, expresses 1 as a combination of the numbers 7 and 9. The equation also gives a way of solving the following problem:

*You have an unlimited supply of water, a drain, a large container, and two jugs which can hold 7 gallons and 9 gallons, respectively. How can you manage to end up with exactly 1 gallon of water in the large container?*

The equation tells you how to do it. Add four of the 9 gallon containers of water into the large container, and then you remove 5 of the 7 gallon containers of water from the large container. You’ll be left with 1 gallon of water in the large container.

### Forget About Your Whiskey, Can You Hold Your Water?

**32.** Solve the previous water problem, except this time you have:

- **a)** 36 gallon and 60 gallon jugs, and you want 12 gallons in the large container.
- **b)** 40 gallon and 70 gallon jugs, and you want 10 gallons in the large container.
- **c)** 450 gallon and 750 gallon jugs, and you want 150 gallons in the large container.

### Are You Ready For Prime Time?

**33.** If you multiply all the prime numbers less than one-million together, will the product be an even number or an odd number? Why?

### Cheaper By The Dozen?

**34.** Find the sum of all the divisors of 6480 that are multiples of 12.

### What Four?

**35.** How many four-digit numbers are multiples of 15, 20, and 25?

### Sister Act.

**36.** Three sisters leave home on the same day. One returns every 5 days, another returns every 4 days, and the third returns every 3 days. How many days until all three sisters meet at home again for the first time?
37. Five sailors were stranded on a desert island, and their only food was coconuts. One day they gathered all the coconuts on the island together, and the next day they would divide them evenly. The first sailor woke up early and gave one coconut to a monkey and hid his fifth of the remaining coconuts. Then the second sailor woke up and gave one coconut to the same monkey and hid his fifth of the remaining coconuts. The third, fourth, and fifth sailors all did the same. Upon arising the next day, one coconut was given to the monkey, and the remaining coconuts were divided equally among the five sailors. What is the smallest starting number of coconuts possible?

\[ N \text{ is the starting number of coconuts, and } R \text{ is the remaining coconuts after the five sailors have done their secret removals.} \]

Or start with a smaller problem: If one sailor gives a coconut to the monkey, takes one-fifth of the remaining coconuts, and then another coconut is given to the monkey and the rest are divided among the five sailors then, the number of remaining coconuts would have to be a multiple of 5: \( \frac{4}{5}(N - 1) = 5k \). So \( N = \frac{5(5k + 1)}{4} + 1 \), which means that \( 5k + 1 \) must be a multiple of 4, and 16 is the smallest possible multiple of 4 that works. This means that the smallest number of coconuts in this case would be 21.

If one sailor gives a coconut to the monkey, takes one-fifth of the remaining coconuts, and another sailor gives a coconut to the monkey, takes one-fifth of the remaining coconuts, and
another coconut is given to the monkey, and the rest are divided among the five sailors then, the number of remaining coconuts would have to be a multiple of 5: \[ \frac{4}{5}(N-1)-1 = 5k \]. So \( N = \frac{125k + 61}{16} \), which means that 125k + 61 must be a multiple of 16, and 1936 is the smallest possible multiple of 16 that works. This means that the smallest number of coconuts in this case would be 121. Keep going!

### A Pirate’s Life For You!

38. Once upon a time, a band of seven pirates seized a treasure chest containing some gold coins (all of equal value). They agree to divide the coins equally among the group, but there were two coins left over. One of the pirates took it upon himself to throw the extra coins overboard to solve the dilemma. Another pirate immediately dived overboard after the sinking coins and was never heard from again. After a few minutes, the six remaining pirates redivided the coins and found that there were three coins left. This time a fight ensued, and one pirate was killed. Finally, the five surviving pirates were able to split the treasure evenly. What was the least possible number of coins in the treasure chest to begin with?

*Hint: If \( x \) is the number of coins, then \( x = 7n + 2 \), \( x = 6m + 5 \), and \( x = 5k + 2 \). This means that \( x - 2 \) is a common multiple of 7 and 5, so \( x - 2 = \{35, 70, 105, 140, 175, \ldots\} \), and therefore \( x = \{37, 72, 107, 142, 177, \ldots\} \). Also \( x - 5 \) is a multiple of 6.*

### Infinitely Many Primes, But None Here!

39. Although there are infinitely many prime numbers, it’s possible to find stretches of positive whole numbers that don’t contain prime numbers that are very large. For example, \( 4! + 2, 4! + 3, 4! + 4 \) is a list of 3 consecutive whole numbers, 26, 27, 28, that don’t contain a prime number. Use this idea to find a list of 10 consecutive whole numbers that don’t contain a prime number.

### Everybody Must Get Stoned.

40. You just bought 504 paving stones for a rectangular outdoor patio. You got a really good deal because they are discontinued, and you won’t be able to get any more. Each stone is a 1 foot square, and you want to use all of them for your patio. What are all the possible dimensions of your patio?

*Hint: What are all the possible factor pairs of 504?*
Prime Filtration.
41. Complete the Sieve of Eratosthenes to determine all the prime numbers less than 300.

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**Don’t Go Rushin’ Through Your Division Problems.**

42. There is a method of division that is similar to Russian Peasant Multiplication that uses doubling and subtraction. Here’s how it works:

For 139 ÷ 12
Keep doubling 12 until the next doubling would exceed 139.

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<td>96 = 8 × 12</td>
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Now subtract these values from largest to smallest if possible from 139 until you are left with a non-negative value less than 12.

139 – 96 = 43, 43 – 24 = 19, 19 – 12 = 7. So 139 ÷ 12 = (8 + 2 + 1) r 7 = 11 r 7. Apply this method to evaluate 185 ÷ 8 and 656 ÷ 58.

---

**Please Hold The Lattice.**

43. Find values for \(a\), \(b\), and \(c\) in the lattice multiplication problem, and also find the product.

{Hint: \(7a = 49\), \(bc = 45\), and \(7b\) must have a ones digit of 5.}

---

**A Prime Time Equation.**

44. Find all whole numbers \(m\) and \(n\) so that the equation \(2^{m+1} + 2^m = 3^{n+2} - 3^n\) is true.