Math 2414 Activity 2 (Due by end of class July 23)

1. a) Use the Intermediate Value Theorem to show that the function \( f(x) = e^x + x \) has a zero on the interval \([-1, 0]\).

\[ \text{Hint: What are the signs of } f(-1) \text{ and } f(0)? \]

b) Use Rolle’s Theorem to show that it has exactly one zero in this interval.

\[ \text{Hint: What’s } f'(x)? \]

2. For what value of \( k \) does the equation \( e^{2x} = k\sqrt{x} \) have exactly one solution?

\[ \text{Hint: From the graph, you can see cases of no solution, one solution, and two solutions. In the case of one solution, the function values and derivative values agree. This leads to the system} \]

\[ \begin{align*}
    e^{2x} &= k\sqrt{x} \\
    2e^{2x} &= \frac{k}{2\sqrt{x}}
\end{align*} \]

\[ \text{system} \]

\[ \text{e}^{2x} \quad \text{Two solutions.} \]

\[ k\sqrt{x} \quad \text{One solution.} \]

\[ k\sqrt{x} \quad \text{No solution.} \]

3. Consider the equation \( e^x = \lambda x \), where \( \lambda \) is a constant.

a) For which values of \( \lambda \) does the equation have a unique solution?

b) For which values of \( \lambda \) does it have no solution?

c) For which values of \( \lambda \) does it have two solutions?

\[ \text{Hint: Check out the graph.} \]
4. Find the **exact** absolute maximum value \((x \text{ and } y \text{ coordinates})\) for the function \(f(x) = 2x - e^x\).

5. For the function \(f(x) = x \ln(2x) - x\) on the interval \([\frac{1}{2e}, \frac{e}{2}]\) Find
   a) the **exact** absolute maximum value \((x \text{ and } y \text{ coordinates})\)
   b) the **exact** absolute minimum value \((x \text{ and } y \text{ coordinates})\)

   \{Hint: Check the critical numbers and endpoints.\}

6. a) Show that the function \(f(x) = e^x - (1 + x)\) has its minimum value at \(x = 0\). What is the minimum value?
   b) Write an inequality between \(e^x\) and \(1 + x\) based on part a).
   c) Replace \(x\) with \(x^2\) in part b) and integrate from 0 to 1 to find upper and lower bounds for \(\int_0^1 e^{x^2} \, dx\).
   d) Since \(e^x = 1 + \int_0^x e'^t \, dt\), \(e^x \geq 1 + \int_0^x (1 + t) \, dt\), so \(e^x \geq 1 + x + \frac{x^2}{2}\). Use this new inequality to find an inequality between \(e^x\) and a cubic polynomial in \(x\), for \(x \geq 0\).
   e) Generalize the previous result to get an inequality between \(e^x\) and an \(n^{th}\) degree polynomial in \(x\), for \(x \geq 0\).
   f) Use the previous results to find \(\lim_{x \to \infty} \frac{e^x}{x^k}\), for any positive integer \(k\).
7. By looking at the graph of the function \( f(x) = \frac{1}{x} \) for \( x > 0 \), find upper and lower bounds on \( \ln(n) \), for \( n \) an integer larger than 1.

So from the figure, you can see that
\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} < \int_{1}^{n} \frac{1}{x} \, dx.
\]
This implies that
\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} < \ln(n)
\]
lower bound

Use the following figure to find an upper bound on \( \ln(n) \).

8. Show that \( f(x) = \frac{x}{e^x - 1} - \ln(1 - e^{-x}) \) is decreasing for \( x > 0 \).

\( \{ \text{Hint: Show that } f'(x) < 0 \text{ for } x > 0. \} \)

9. Let \( f(x) = \frac{\ln x}{1 + (\ln x)^2} \) for \( x \) in \( (0, \infty) \).
a) By rewriting \( f \) as \( f(x) = \frac{1}{\ln x + \ln x} \), find \( \lim_{{x \to 0^+}} f(x) \).

b) By rewriting \( f \) as \( f(x) = \frac{1}{\ln x + \ln x} \), find \( \lim_{{x \to \infty}} f(x) \).

c) Find the maximum value of \( f \). (\( x \) and \( y \) coordinates)

d) Find the minimum value of \( f \). (\( x \) and \( y \) coordinates)

10. Calculate \( \lim_{{n \to \infty}} \left[ \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right] \) by rewriting it as \( \lim_{{n \to \infty}} \frac{1}{n} \left[ \frac{1}{1/n} + \frac{1}{1+2/n} + \cdots + \frac{1}{1+n/n} \right] \), and identifying it as a definite integral.

11. Apply the Mean Value Theorem to the function \( f(x) = \ln(1 + x) \) for \( x > -1 \) to show that \( \frac{x}{1+x} \leq \ln(1+x) \leq x ; x > -1 \).

\( \{ f(x) - f(0) = f'(c) \cdot x \implies \ln(1+x) = \frac{x}{1+c} \text{ for some } c \text{ between } x \text{ and } 0. \} \)

\textbf{Case I:} If \( -1 < x \leq 0 \), then \( \frac{1}{1+c} > 1 \) and \( \frac{x}{1+c} \leq x \). Also, \( \frac{1}{1+c} < \frac{1}{1+x} \), so \( \frac{x}{1+c} < \frac{x}{1+x} \).

Now you take care of Case II, where \( x > 0 \).

12. Suppose that \( f'(x) = f(x) \) for all \( x \) and \( f(0) = 0 \). Then \( e^{-x}f'(x) - e^{-x}f(x) = 0 \), which implies from the product rule that \( \left[ e^{-x}f(x) \right]' = 0 \), and therefore that \( e^{-x}f(x) = C \).

Plugging in 0 for \( x \) yields \( C = 0 \), and we can conclude that \( f \) must be the zero function. So the only solution of \( f'(x) = f(x) \) with \( f(0) = 0 \) is \( f(x) \equiv 0 \). Find as many non-constant solutions of \( f'(x) \leq f(x) \) with \( f(0) = 0 \) as you can.

\{\text{Hint: Here’s one: } f(x) = \begin{cases} x^2; & x \leq 0 \\ 0; & x \geq 0 \end{cases}\}
13. Show that \( \ln \left( \frac{a+b}{2} \right) > \frac{\ln a + \ln b}{2} \) for \( a, b > 0, a \neq b \).

Let \( L(x) = \frac{2}{a+b} \left( x - \frac{a+b}{2} \right) + \ln \left( \frac{a+b}{2} \right) \) be the tangent line to the graph of \( \ln x \) at \( x = \frac{a+b}{2} \). Since \( (\ln x)'' = -\frac{1}{x^2} < 0 \), we have that \( L(x) > \ln x \) for \( x \neq \frac{a+b}{2} \). This means that \( \frac{L(a) + L(b)}{2} > \frac{\ln a + \ln b}{2} \). If you work out the left side of the inequality, you’ll get the result.

14. Let \( f(x) = \frac{\ln x}{x} - \frac{1}{e}; x > 0 \).

a) Find the maximum value of \( f(x) \) \((x \text{ and } y \text{ coordinates})\).

b) Find all positive numbers \( x \) so that \( x^e < e^x \).

\{Hint: If \( x^e < e^x \), then \( e \ln x < x \Rightarrow \frac{\ln x}{x} < \frac{1}{e} \Rightarrow \frac{\ln x}{x} - \frac{1}{e} < 0 \Rightarrow f(x) < 0.\}\}

15. For \( t \geq 1, \frac{1}{t} \leq \frac{1}{\sqrt{t}} \), so \( 0 \leq \ln x = \int_1^x \frac{1}{t} \, dt \leq \int_1^x \frac{1}{\sqrt{t}} \, dt = 2 \left( \sqrt{x} - 1 \right) \).
a) Use the previous result to find \( \lim_{x \to \infty} \frac{\ln x}{x} \). \( \text{[Hint: Give it a squeeze.]} \)

b) Use the previous part to find \( \lim_{x \to 0^+} x \ln x \). \( \text{[Hint: Let } x = \frac{1}{t} \text{ in } x \ln x, \text{ and let } t \to 0^+.} \)

16. a) Show that if \( b > a \) and \( f \) is a continuous function with \( f(x) + f(a + b - x) \neq 0 \) for \( a \leq x \leq b \), then
\[
\int_{a}^{b} \frac{f(x)}{f(x) + f(a + b - x)} \, dx = \int_{a}^{b} \frac{f(a + b - x)}{f(x) + f(a + b - x)} \, dx.
\]
(\( \text{[Hint: Use the substitution } u = a + b - x \text{ in the integral on the left.]} \)

b) Find the actual value of the previous integral.
(\( \text{[Hint: If } I = \int_{a}^{b} \frac{f(x)}{f(x) + f(a + b - x)} \, dx \text{, then from part a), } I = \int_{a}^{b} \frac{f(a + b - x)}{f(x) + f(a + b - x)} \, dx. \]
So add these two equations and solve for } I. \)

c) Use the previous part to evaluate \( \int_{2}^{4} \frac{\sqrt{\ln(9 - x)}}{\sqrt{\ln(9 - x)} + \sqrt{\ln(x + 3)}} \, dx. \)

17. Consider the function \( f(x) = \frac{1}{1 + e^x}. \) Determine the following limits:

a) \( \lim_{x \to \infty} f(x) \) 

b) \( \lim_{x \to \infty} f(x) \)

c) \( \lim_{x \to 0^+} f(x) \)

d) \( \lim_{x \to 0^+} f(x) \)

18. Consider the function \( f(x) = \frac{e^x}{1 + e^x}. \) Determine the following limits:

a) \( \lim_{x \to \infty} f(x) \)

b) \( \lim_{x \to \infty} f(x) \)

c) \( \lim_{x \to 0^+} f(x) \)

d) \( \lim_{x \to 0^+} f(x) \)

19. Find the following limits:

a) \( \lim_{a \to 0^+} \log_a 2 \)

b) \( \lim_{a \to 1^+} \log_a 2 \)

c) \( \lim_{a \to 1^+} \log_a 2 \)

d) \( \lim_{a \to \infty} \log_a 2 \)
(\( \text{[Hint: } \log_a 2 = \frac{\ln 2}{\ln a}. \)

20. Find the point on the curve \( y = x + \ln(x^2 + 1) \) at which both the first and second derivatives are equal to zero. Is this an inflection point for the curve?

21. Suppose that \( f \) is a function with the property that for all positive real numbers \( x \) and \( y \),
\( f(xy) = f(x) + f(y). \)

a) Find \( f(1). \)

b) Show that if \( f'(1) \) exists, then \( f'(x) \) exists for all positive \( x. \)
(\( \text{[Difference Quotient.]} \)
22. Consider the function \( f(x) = x^n e^x \), where \( n \) is an even integer greater than zero. What’s the connection between the \( x \)-coordinate of the local maximum of \( f \) and the \( x \)-coordinates of the two inflection points of \( f \)?

23. If \( \log_b (3^b) = \frac{b}{2} \), then what’s the value of \( b \)?

24. Consider the function \( f(x) = \frac{1}{1-e^x} + \frac{1}{1-e^{-x}} \).
   a) Find \( \lim_{x \to 0^+} f(x) \).
   b) Find \( \lim_{x \to 0^-} f(x) \).
   c) Find \( \lim_{x \to 0} f(x) \).

   \{Hint: Combine the two fractions in the formula for \( f \) into one.\}

25. Differentiate the following functions using logarithmic differentiation:
   a) \( f(x) = x^{(x+\sin x)} \)
   b) \( f(x) = (\sin x)^{\cos x} \)
   c) \( f(x) = x^{(x^2)} \)

26. Find the intervals of increase and decrease for the following functions:
   a) \( f(x) = \log_x 2 \)
   b) \( f(x) = \log_x \frac{1}{2} \)

27. In the function \( f(x) = x^x \) both the base and the exponent are variable. Its derivative can be calculated using logarithmic differentiation. What if the number of \( x \)'s in the exponent is also variable? Specifically, define the continued exponentiation function, \( \text{cont}(x) = x^{x^{x^{x^{\cdots}}}} \), where there is \( x \) number of \( x \)'s in the exponent. For example, \( \text{cont}(3) = 3^{3^3} = 3^{27} = 3^{7,625,597,484,987} \), which has over 3.6 billion digits. So far the definition of \( \text{cont}(x) \) makes sense only for positive integer values. How could we define \( \text{cont} \left( \frac{3}{2} \right) \)?
Perhaps \[ \text{cont} \left( \frac{3}{2} \right) = \left( \frac{3}{2} \right)^{\left( \frac{3}{2} \right)^{\frac{1}{2}}} \], \[ \text{cont} \left( \frac{1}{3} \right) = \left( \frac{1}{3} \right)^{\left( \frac{1}{3} \right)^{\frac{1}{3}}} \], and \[ \text{cont} \left( \sqrt{2} \right) = \left( \sqrt{2} \right)^{\left( \sqrt{2} \right)^{\frac{1}{2}}} \]. If you see the pattern, then expand the following:

a) \[ \text{cont} \left( \frac{5}{2} \right) \]  

b) \[ \text{cont} \left( \frac{10}{3} \right) \]  

c) \[ \text{cont} \left( \sqrt{5} \right) \]

28. Express the following in terms of \( \ln 2 \) and/or \( \ln 5 \).

a) \( \ln 10 \)  
b) \( \ln \frac{1}{2} \)  
c) \( \ln \frac{1}{5} \)  
d) \( \ln 25 \)  
e) \( \ln \sqrt{2} \)  
f) \( \ln \sqrt[3]{5} \)  
g) \( \ln \frac{1}{20} \)  
h) \( \ln 2^{12} \)

29. Find the maximum value of \( f(x) = x^\frac{1}{2} \) on the interval \( (0, \infty) \).

30. a) Show that as \( x \) decreases towards 1, \( f(x) = \frac{\ln x}{x-1} \) increases. In other words, show that \( f'(x) < 0 \) for \( x > 1 \).

\[ \text{Hint: } f'(x) = \frac{x-1-x\ln x}{x(x-1)^2} \]  
If you can show that the numerator is negative for \( x > 1 \), you’ll have the result. For \( g(x) = x-1-x\ln x \), \( g(1) = 0 \), now see if you can show that \( g'(x) < 0 \) for \( x > 1 \).

b) Replace \( x \) with \( 1 + \frac{1}{x} \) to show that \( h(x) = \left( 1 + \frac{1}{x} \right)^x \) is increasing for \( x > 0 \).

\[ \text{Hint: } \ln \left( 1 + \frac{1}{x} \right) = \ln \left( \left( 1 + \frac{1}{x} \right)^x \right), \text{ and } f(x) \text{ is increasing if and only if } e^{f(x)} \text{ is increasing.} \]

31. Suppose that \( f \) is a continuously differentiable function such that \( f(x) + f'(x) \leq 1 \) for all \( x \), and \( f(0) = 0 \). What is the largest possible value of \( f(1) \)?

\[ \text{Hint: Multiply the inequality by } e^x. \text{ Recognize the left side as } (e^x f(x))', \text{ and integrate from 0 to 1.} \]

32. Let \( f(x) = e^x - kx \) with \( k > 0 \).

a) Show that \( f \) has a local minimum at \( x = \ln k \).

b) Find the value of \( k \) for which this local minimum value of \( f \) is as large as possible.

33. Show that the equation \( e^{2x} + \cos x + x = 0 \) has exactly one solution.

\[ \text{Hint: Check out } f(0) \text{ and } f(-\pi). \text{ Also examine } f'(x), \text{ and consider Rolle’s Theorem.} \]
34. Find the **exact** extrema \((x \text{ and } y \text{ coordinates})\) for the function \(f(x) = \arcsin(x) - 2x\).

35. You are sitting in a classroom next to the wall looking at the whiteboard at the front of the room. The whiteboard is 12 ft. long and starts 3 ft. from the wall you are sitting next to.

Your viewing angle is given by \(\alpha(x) = \cot^{-1}\left(\frac{x}{15}\right) - \cot^{-1}\left(\frac{x}{3}\right)\). How far from the front of the classroom should you sit to maximize your viewing angle?

There is a unique circle that passes through the two points on the left and right edges of the white board at your eye level and your eye. What’s special about the point on the circle at your eye?

---

Evaluate the following integrals:

36. \(\int \frac{dx}{\sqrt{1 - 4x^2}}\)

37. \(\int \frac{dx}{17 + x^2} \quad \{\text{Hint: } 17 = \left(\sqrt{17}\right)^2\} \)

38. \(\int_{-2}^{2} \frac{dt}{4 + 3t^2} \quad \{\text{Hint: Factor the 3.}\}

39. \(\int_{0}^{\ln\sqrt{3}} \frac{e^x \, dx}{1 + e^{2x}} \quad \{\text{Hint: Let } u = e^x.\} \)
40. ∫₀⁻¹ 6dt/√(3 - 2t - t²) \{Complete the square.\}  

41. ∫ dy/y² + 6y + 10 \{Complete the square.\}  

42. ∫ e^{cos⁻¹ x} dx/√(1 - x²) \{Hint: Let u = cos⁻¹ x.\}  

43. For 0 ≤ x ≤ 1, 0 < 4 - 2x² ≤ 4 - x² - x³ ≤ 4 - x².  

So by taking square roots, you get that 0 < √(4 - 2x²) ≤ √(4 - x² - x³) ≤ √(4 - x²).  

And by taking reciprocals, you get \(\frac{1}{\sqrt{4 - x^2}} \leq \frac{1}{\sqrt{4 - x^2 - x^3}} \leq \frac{1}{\sqrt{4 - 2x^2}}\).  

Integrate this inequality from 0 to 1 and find upper and lower bounds on the value of \(\int_0^1 \frac{1}{\sqrt{4 - x^2 - x^3}} \, dx\).  

44. a) Show that \(\int_0^1 \frac{x^4(1-x)^4}{1+x^2} \, dx = \frac{22}{7} - \pi\), using \(x^4(1-x)^4 = x^8 - 4x^7 + 6x^6 - 4x^5 + x^4\) and completing the long division  

\[
\frac{x^6}{x^2+1) x^8 - 4x^7 + 6x^6 - 4x^5 + x^4} - (x^8 + x^6) \quad -4x^7 + 5x^6 - 4x^5 + x^4
\]

b) For 0 ≤ x ≤ 1, \(\frac{1}{2} \leq \frac{x^4(1-x)^4}{1+x^2} \leq \frac{x^4(1-x)^4}{1} \). This means that  

\(\frac{1}{2} \int_0^1 x^4(1-x)^4 \, dx \leq \int_0^1 \frac{x^4(1-x)^4}{1+x^2} \, dx \leq \int_0^1 x^4(1-x)^4 \, dx\). Evaluate \(\int_0^1 x^4(1-x)^4 \, dx\) to get an inequality of the form \(m \leq \frac{22}{7} - \pi \leq M\), and see how close \(\frac{22}{7}\) is to \(\pi\).
45. Find the *exact* inflection point of \( f(x) = (x + 1)\tan^{-1} x \) (x and y coordinates).

*Hint: Check out \( f''(x) \).*

46. Two towns A and B are 8 miles apart. A third town C is located 5 miles from both A and B. If the point P, equidistant from A and B is such that the sum of the distances PA, PB, and PC is the least possible, how far is the point P from C?

The sum of the lengths in terms of the angle \( \theta \) is given by

\[
L(\theta) = 3 + 8\sec \theta - 4\tan \theta \quad ; \quad 0 \leq \theta \leq \tan^{-1}\left(\frac{3}{4}\right).
\]
47. A 27 ft. ladder is placed vertically against an 8 ft. high fence. The lower end of the ladder is then pulled directly away from the fence. If the ladder is kept in contact with the top of the fence, what is the greatest horizontal distance the ladder ever projects beyond the fence?

The horizontal projection as a function of $\theta$ is given by

$$P(\theta) = 27 \cos \theta - 8 \cot \theta; \sin^{-1}(\frac{8}{27}) \leq \theta \leq \frac{\pi}{2}.$$ 

48. Apply the Mean Value Theorem to the function $f(x) = \tan^{-1} x$ for $x \geq 0$ to show that

$$\frac{x}{1 + x^2} \leq \tan^{-1} x \leq x; x \geq 0.$$ 

$$\{ f(x) - f(0) = f'(c)x \Rightarrow \tan^{-1} x = \frac{x}{1+c^2} \text{ for some } c \text{ between } x \text{ and } 0. \}$$

49. Use differentiation to show that $\tan^{-1} x + \tan^{-1}(\frac{1}{x}) = \begin{cases} \frac{\pi}{2}; x > 0 \\ -\frac{\pi}{2}; x < 0 \end{cases}.$

50. Find the angle $\theta$ in the following figure:

$$\{ \text{Hint: } \tan(\theta + \frac{\pi}{6}) = \frac{3 - \sqrt{3} + 3\tan(\frac{\pi}{6})}{3}. \}$$
51. Find the value of \( \tan^{-1}\left(\frac{a}{x}\right) + \tan^{-1}\left(\frac{b}{x}\right) \) when \( x \) is the \( x \)-coordinate of the maximum value of \( f(x) = \tan^{-1}\left(\frac{a}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right) \), for \( a > b > 0 \).

52. a) Use the fact that for \( 0 \leq x \leq \frac{1}{2} \), \( 1 < \frac{1}{\sqrt{1-x^4}} < \frac{1}{\sqrt{1-x^2}} \), to find upper and lower bounds for the integral \( \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^4}} \, dx \).

b) Use the fact that for \( 0 \leq x \leq 1 \), \( \frac{1}{2} < \frac{1}{\sqrt{4-x^2+x^4}} < \frac{1}{\sqrt{4-x^2}} \), to find upper and lower bounds for the integral \( \int_0^1 \frac{1}{\sqrt{4-x^2+x^4}} \, dx \).

53. Evaluate \( \int \frac{1}{x\sqrt{x^{10}+x^5-1}} \, dx \). Factor \( x^{10} \) from the radicand to get \( \int \frac{1}{x^6\sqrt{1+x^{-5}-(x^{-5})^2}} \, dx \), let \( u = x^{-5} \), and then complete the square.

54. Use differentiation to show that \( \tan^{-1}\left(\frac{x-1}{x+1}\right) = \tan^{-1} x - \frac{\pi}{4} \) is an identity.

55. Show that \( f(x) = \tan^{-1}\left(\frac{x+2}{x}\right) + \tan^{-1}(x+1) \) is a constant function.

56. Show that \( \tan^{-1} x < \frac{\pi}{4} + \frac{1}{2} \ln x \) for \( x > 1 \).

\{\text{Hint: For } f(x) = \tan^{-1} x - \frac{\pi}{4} - \frac{1}{2} \ln x, \ f(1) = 0. \ See if you can show that } f'(x) < 0 \text{ for } x > 1 \text{ to get the result.}\}
57. Find all functions \( f \) such that \( f' \) is continuous and

\[
\left[ f(x) \right]^2 = 100 + \int_0^x \left\{ \left[ f(t) \right]^2 + \left[ f'(t) \right]^2 \right\} \, dt \quad \text{for all real } x.
\]

\{Hint: Differentiate both sides and use \( (a - b)^2 = a^2 - 2ab + b^2 \).\}

58. Find all continuous functions \( f \) so that \( f(x) = \int_0^x f(t) \, dt + 1 \).

\{Hint: Differentiate the integral equation and then use the integral equation to determine an initial condition.\}

59. Suppose that \( f \) is a continuous, non-constant function on \([0,1]\) with \( \int_0^1 f(x) \, dx = 0 \). Consider the function \( g(a) = \int_0^1 e^{af(x)} \, dx \).

a) Find the value of \( g(0) \).

b) Find the value of \( g'(0) \).

\{Hint: According to a theorem of Leibniz, \( g'(a) = \int_0^1 \frac{d}{da} [e^{af(x)}] \, dx = \int_0^1 f(x)e^{af(x)} \, dx \).\}

c) Show that \( g''(a) > 0 \) for all \( a \).

\{Hint: \( g''(a) = \int_0^1 \frac{d}{da} [f(x)e^{af(x)}] \, dx = \int_0^1 [f(x)]^2 e^{af(x)} \, dx \).\}

d) What’s the minimum value of \( g \)?

\{Hint: The graph of \( g \) is concave-up.\}

60. a) Use the fact that \( \frac{x^2}{x^2 + 1} < 1 \) to show that \( \int_0^1 \frac{x^2}{x^2 + 1} \, dx < 1 \).

b) Use the fact that \( \int_0^1 \frac{x^2}{x^2 + 1} \, dx = \int_0^1 \left( \frac{x^2 + 1}{x^2 + 1} - \frac{1}{x^2 + 1} \right) \, dx \) to find the exact value of \( \int_0^1 \frac{x^2}{x^2 + 1} \, dx \).
61. If \( f(x) = \int_{1}^{x} \frac{\ln t}{1 + t} \, dt \), then find the value of \( f(\sqrt{e}) + f\left(\frac{1}{\sqrt{e}}\right) \).

\[
\{\text{Hint: } f(\sqrt{e}) + f\left(\frac{1}{\sqrt{e}}\right) = \int_{1}^{\sqrt{e}} \frac{\ln t}{1 + t} \, dt + \int_{1}^{\frac{1}{\sqrt{e}}} \frac{\ln t}{1 + t} \, dt. \text{ Let } u = \frac{1}{t} \text{ in the second integral to get}
\]

\[
\int_{1}^{\sqrt{e}} \frac{\ln t}{1 + t} \, dt = \int_{1}^{\sqrt{e}} \frac{\ln u}{u^2 + u} \, du. \text{ Now rewrite the sum of the integrals as}
\]

\[
\int_{1}^{\sqrt{e}} \frac{1}{t^2 + t} \, dt, \text{ and simplify the integrand.}\]

62. Find the value of \( I = \int_{\frac{1}{2}}^{2} \frac{\ln x}{1 + x^2} \, dx \).

\[
\{\text{Hint: Let } u = \frac{1}{x}.\}
\]

63. Show that \( \ln x \) is not a rational function restricted to \((0, \infty)\).

\[
\{\text{Hint: Suppose that } \ln x = \frac{N(x)}{D(x)}, \text{ where } N \text{ and } D \text{ are polynomial functions with no}
\]

common factors. Differentiation leads to

\[
\frac{1}{x} = \frac{D(x)N'(x) - D'(x)N(x)}{D^2(x)} \Rightarrow D^2(x) = x \left[ D(x)N'(x) - D'(x)N(x) \right]. \text{ This}
\]

means that \( x \) must be a factor of \( D(x) \), and therefore, \( D(x) = x^kD_k(x) \) with \( k \geq 1 \)

and \( D_k(0) \neq 0 \). Substituting this into the previous equation leads to

\[
x^{2k}D_k^2(x) = x^{k+1}D_k(x)N'(x) - x^{k+1}D_k'(x)N(x) - kx^kD_k(x)N(x), \text{ and dividing by } x^k
\]

leads to \( x^kD_k^2(x) = xD_k(x)N'(x) - xD_k'(x)N(x) - kD_k(x)N(x) \). Plugging in zero

for \( x \) leads to \( kD_k(0)N(0) = 0 \Rightarrow N(0) = 0 \). What does this imply is a factor of

\( N(x) \), and what can you conclude?\}

64. Use differentiation to verify the identity \( \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sin^{-1} x \) for \(-1 < x < 1\).

65. Use differentiation to verify the identity \( \tan^{-1}\left(\frac{1-x}{\sqrt{1-x^2}}\right) = \frac{\pi}{4} - \frac{1}{2}\sin^{-1} x \) for \(-1 < x < 1\).