Determinate and Indeterminate Limit Forms

Some limits can be determined by inspection just by looking at the form of the limit – these predictable limit forms are called determinate. Other limits can’t be determined just by looking at the form of the limit and can only be determined after additional work is done – these unpredictable limit forms are called indeterminate.

Determinate Limit Forms:

Assuming that the functions involved in the limit are defined:

1. The limit forms \( \frac{a}{\pm \infty} \), for any number \( a \), and \( \frac{\text{bounded}}{\pm \infty} \) result in a limit of 0.

2. The limit form \( \infty \cdot \infty \) results in a limit of \( \infty \).

3. The limit form \( (\pm \infty)^{-\infty} \) results in a limit of 0.

4. The limit form \( a^\infty \), for \(-1 < a < 1\), results in a limit of 0.

5. The limit form \( a^\infty \), for \( a > 1 \), results in a limit of \( \infty \).

6. The limit form \( \infty^a \), for \( a > 0 \), results in a limit of \( \infty \).

7. The limit form \( (\pm \infty)^a \), for \( a < 0 \), results in a limit of 0.

8. The limit forms \( \frac{\pm \infty}{0^+} \), \( \frac{\pm \infty}{a} \), and \( \pm \infty \cdot a \), for \( a > 0 \), result in a limit of \( \pm \infty \).

9. The limit forms \( \frac{\pm \infty}{0^-} \), \( \frac{\pm \infty}{a} \), and \( \pm \infty \cdot a \), for \( a < 0 \), result in a limit of \( \mp \infty \).

10. The limit forms \( \infty \cdot (-\infty) \) and \( -\infty \cdot -\infty \) result in a limit of \( -\infty \).

11. The limit forms \( \infty + \infty \), \( \infty \cdot \infty \), and \( (-\infty) \cdot (-\infty) \) result in a limit of \( \infty \).

12. The limit forms \( a \cdot 0 \), for any number \( a \), and \( (\text{bounded}) \cdot 0 \) result in a limit of 0.

13. The limit forms \( \pm \infty \pm a \), for any number \( a \), and \( \pm \infty \pm (\text{bounded}) \) result in a limit of \( \pm \infty \).

14. The limit form \( a^0 \), for \( a > 0 \), results in a limit of 1.
Indeterminate Limit Forms:

1. $1^\pm\infty$

   \[
   \lim_{x \to \infty} (1 + \frac{1}{x})^{-x^2} = 0
   \]
   \[
   \lim_{x \to \infty} (1 + \frac{\ln a}{x})^x = a, \text{ for } a > 0
   \]
   \[
   \lim_{x \to \infty} (1 + \frac{1}{x})^x = \infty
   \]
   \[
   \lim_{x \to \infty} (1 + \frac{\sin x}{x})^x = DNE
   \]

   As you can see, this limit form can result in all limits from 0 to $\infty$, and even $DNE$.

2. $0^0$

   \[
   \lim_{x \to 0^-} \frac{x}{x^2} = -\infty
   \]
   \[
   \lim_{x \to 0^+} \frac{ax}{x} = a, \text{ for any number } a
   \]
   \[
   \lim_{x \to 0^-} \frac{x}{x^2} = \infty
   \]
   \[
   \lim_{x \to 0^+} \frac{x}{x^2} = DNE, \quad \lim_{x \to 0} \frac{x \sin \left( \frac{1}{x} \right)}{x} = DNE
   \]

   As you can see, this limit form can result in all limits from $-\infty$ to $\infty$, and even $DNE$. 
3. \( \frac{\pm \infty}{\pm \infty} \)

\[
\lim_{x \to \pm \infty} \frac{x^2}{x} = -\infty
\]

\[
\lim_{x \to \pm \infty} \frac{ax}{x} = a, \text{ for } a \neq 0
\]

\[
\lim_{x \to \pm \infty} \frac{x}{x^2} = 0
\]

\[
\lim_{x \to \pm \infty} \frac{x^2}{x} = \infty
\]

\[
\lim_{x \to \pm \infty} \frac{2x + x\sin x}{x} = DNE
\]

As you can see, this limit form can result in all limits from \(-\infty\) to \(\infty\), and even \(DNE\).

4. \( \pm \infty \cdot 0 \)

\[
\lim_{x \to \pm \infty} \left( x^2 \cdot \frac{1}{x} \right) = -\infty
\]

\[
\lim_{x \to \pm \infty} \left( x \cdot \frac{a}{x} \right) = a, \text{ for any number } a
\]

\[
\lim_{x \to \pm \infty} \left( x^2 \cdot \frac{1}{x} \right) = \infty
\]

\[
\lim_{x \to \pm \infty} \left( 2x + x\sin x \right) \cdot \frac{1}{x} = DNE
\]

As you can see, this limit form can result in all limits from \(-\infty\) to \(\infty\), and even \(DNE\).
5. \( \infty^0 \)

\[
\lim_{x \to \infty} \left( e^x \right)^{\left( \frac{1}{\ln x} \right)} = 0
\]

\[
\lim_{x \to \infty} \left[ \left( \frac{1}{a} \right)^x \right]^{\left( -\frac{1}{x} \right)} = a, \text{ for } 0 < a < 1
\]

\[
\lim_{x \to \infty} x^{\frac{1}{3}} = 1
\]

\[
\lim_{x \to \infty} \left( a^x \right)^{\frac{1}{a}} = a, \text{ for } a > 1
\]

\[
\lim_{x \to 0} \left( x^x \right)^{\frac{1}{x}} = \infty
\]

\[
\lim_{x \to \infty} \left[ \left( 3 + \sin x \right)^x \right]^{\frac{1}{x}} = DNE
\]

As you can see, this limit form can result in all limits from 0 to \( \infty \), and even \( DNE \).

6. \( 0^0 \)

\[
\lim_{x \to 0^+} x^{\left( \frac{1}{\sqrt{x^2}} \right)} = 0
\]

\[
\lim_{x \to 0^+} x^{\ln a} = a, \text{ for } a > 0
\]

\[
\lim_{x \to 0^+} \left[ \left( \frac{1}{2} \right)^x \right]^{\left( -x \right)} = \infty
\]

\[
\lim_{x \to 0^+} x^x = DNE, \quad \lim_{x \to 0^+} \left\{ x^{\left[ \sin^2 \left( \frac{1}{x} \right) \right]^\frac{1}{x}} \right\} = DNE
\]

As you can see, this limit form can result in all limits from 0 to \( \infty \), and even \( DNE \).
$\lim_{x \to \infty} \left( x - x^2 \right) = -\infty$

$\lim_{x \to \infty} \left( x + a \right) - x = a$, for any number $a$

$\lim_{x \to \infty} \left( x^2 - x \right) = \infty$

$\lim_{x \to \infty} \left( x + \sin x \right) - x = DNE$

As you can see, this limit form can result in all limits from $-\infty$ to $\infty$, and even $DNE$.

**L’Hôpital’s Rule Guidelines:**

<table>
<thead>
<tr>
<th>Type of indeterminate form</th>
<th>Apply L’Hôpital’s Rule to</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\lim \frac{f(x)}{g(x)} = 0$ or $\lim \frac{f(x)}{g(x)} = \pm\infty$</td>
<td>$\lim \frac{f(x)}{g(x)}$</td>
</tr>
<tr>
<td>or $\lim \frac{f(x)}{g(x)} = \text{whatever}$</td>
<td>$\pm\infty$</td>
</tr>
<tr>
<td>2. $\lim f(x)g(x) = \pm\infty \cdot 0$</td>
<td>$\lim \frac{f(x)}{g(x)}$ or $\lim \frac{1}{f(x)}$</td>
</tr>
<tr>
<td>3. $\lim f(x)^{g(x)} = 1^{\pm\infty}$, $\lim f(x)^{g(x)} = 0^0$, $\lim f(x)^{g(x)} = \infty^0$</td>
<td>$\lim \frac{\ln f(x)}{g(x)}$ or $\lim \frac{g(x)}{\ln f(x)}$</td>
</tr>
<tr>
<td><strong>But remember to exponentiate to get the original limit.</strong></td>
<td></td>
</tr>
<tr>
<td>4. $\lim \left[ f(x) - g(x) \right] = \infty - \infty$</td>
<td>$\lim \left[ \frac{g(x)}{f(x)} - \frac{g(x)}{f(x)} \right]$</td>
</tr>
<tr>
<td>5. $\frac{f(x)}{g(x)} = \frac{0}{\text{whatever}}$</td>
<td>$\lim \frac{1}{g(x)}$</td>
</tr>
</tbody>
</table>
Warning: If the zeros of \( g'(x) \) accumulate at \( a \), then it might be the case that \( \lim_{x \to a} \frac{f'(x)}{g'(x)} \)
appears to exist, but \( \lim_{x \to a} \frac{f(x)}{g(x)} \neq \lim_{x \to a} \frac{f'(x)}{g'(x)} \).

Example:

\[
\lim_{x \to \infty} \frac{x + \cos x \sin x}{e^{\sin x} (x + \cos x \sin x)}
\]
has the form \( \frac{\infty}{\infty} \), and the limit doesn’t exist. However,

\[
\lim_{x \to \infty} \frac{(x + \cos x \sin x)'}{e^{\sin x} (x + \cos x \sin x)'} = \lim_{x \to \infty} \frac{e^{\sin x} (1 - \sin^2 x + \cos^2 x) + \cos x e^{\sin x} (x + \cos x \sin x)}{2 \cos^2 x e^{\sin x} \cos^2 x + \cos x e^{\sin x} (x + \cos x \sin x)}
\]

Since \( \lim_{x \to \frac{(2n-1)\pi}{2}} \frac{2 \cos^2 x}{2 e^{\sin x} \cos^2 x + \cos x e^{\sin x} (x + \cos x \sin x)} = 0 \), the discontinuities in the function \( \frac{2 \cos^2 x}{2 e^{\sin x} \cos^2 x + \cos x e^{\sin x} (x + \cos x \sin x)} \) can be removed to yield the continuous function

\[
\begin{cases}
2 \cos x \\
2 e^{\sin x} \cos x + e^{\sin x} (x + \cos x \sin x); \cos x \neq 0 \\
0; \cos x = 0
\end{cases}
\]

So a careless cancelation of \( \cos x \) between the numerator and denominator would lead you to believe that \( \lim_{x \to \infty} \frac{x + \cos x \sin x}{e^{\sin x} (x + \cos x \sin x)} = 0 \), when it actually doesn’t exist.