Math 2412 Activity 3 (Due by EOC Nov. 11)

Graph the following exponential functions by modifying the graph of \( f(x) = 2^x \). Find the range of each function.

1. \( g(x) = 2^{x+2} \)
2. \( g(x) = 2^x + 2 \)
3. \( g(x) = 2^{x^2} - 1 \)
4. \( g(x) = 2^{-x} \)
5. \( g(x) = -2^x \)
6. \( g(x) = -2^{-x} \)

Find a formula for the exponential function whose graph is given

7. 

8. 

Write each equation in its equivalent exponential form.

9. \( 6 = \log_2 64 \)
10. \( 2 = \log_9 x \)
11. \( \log_5 125 = y \)

Write each equation in its equivalent logarithmic form.

12. \( 5^4 = 625 \)
13. \( 5^{-3} = \frac{1}{125} \)
14. \( 8^y = 300 \)

Simplify the following expressions.

15. \( \log_7 49 \)
16. \( \log_3 27 \)
17. \( \log_6 \frac{1}{6} \)
18. \( \log_3 \frac{1}{9} \)
19. \( \log_6 \sqrt{6} \)
20. \( \log_3 \frac{1}{\sqrt{3}} \)
21. \( \log_8 1 \)
22. \( \log_6 1 \)
23. \( 7^{\log_7 23} \)
24. \( \log_5 (\log_2 32) \)
25. \( \log_4 \left[ \log_3 (\log_2 8) \right] \)
26. \( \frac{\log_3 81 - \log_2 1}{\log_2 \sqrt{8} - \log_{10} .001} \)
27. \( \log_{\frac{1}{2}} 16 + \log_{\frac{1}{3}} 9 \)  
28. \( \log_{16} 4 - \log_{27} 3 \)  
29. \( \log_{7} \frac{1}{2401} + \log_{8} \frac{1}{64} + \log_{2} \frac{1}{64} \)

30. \( \log_{\frac{1}{2}} \frac{1}{4} - \log_{\frac{1}{3}} \frac{1}{27} \)

Graph the following logarithmic functions by modifying the graph of \( f(x) = \log_{2} x \). Find the domain of each function.

31. \( g(x) = \log_{2} (x + 2) \)  
32. \( g(x) = 2 + \log_{2} x \)  
33. \( g(x) = -2\log_{2} x \)  
34. \( g(x) = \log_{2} (-x) \)

Expand each expression as much as possible, and simplify whenever possible.

35. \( \log_{8} (64 \cdot 7) \)  
36. \( \log_{9} (9x) \)  
37. \( \log_{10} (10,000x) \)  
38. \( \log_{9} \left( \frac{9}{x} \right) \)

39. \( \log_{b} x^{7} \)  
40. \( \log_{10} M^{-8} \)  
41. \( \log_{2} \sqrt[3]{x} \)  
42. \( \log_{b} (xy^{3}) \)

43. \( \log_{5} \left( \frac{\sqrt{x}}{25} \right) \)  
44. \( \log_{10} \left[ \frac{100x^{3} \sqrt[3]{5 - x}}{3(x + 7)^{2}} \right] \)

Compress each expression as much as possible, and simplify whenever possible.

45. \( \log_{10} 250 + \log_{10} 4 \)  
46. \( \log_{3} 405 - \log_{3} 5 \)  
47. \( \log_{2} x + 7\log_{2} y \)

48. \( 7\log_{10} x - 3\log_{10} y \)  
49. \( \frac{1}{3} (\log_{4} x - \log_{4} y) + 2\log_{4} (x + 1) \)

Use a calculator to evaluate the following logarithms to four places.

50. \( \log_{6} 17 \)  
51. \( \log_{16} 57.2 \)  
52. \( \log_{\pi} 400 \)

53. Simplify \( 10^{\log_{10} 8x^{2} - \log_{10} 2x^{2}} \), for \( x > 0 \).

54. Find the value of \( x \) for \( a > 0 \) so that \( \log_{2} \left( \log_{3} \left( \log_{a} x \right) \right) = 1 \)

Solve the following exponential equations.

55. \( 3^{x} = 81 \)  
56. \( 3^{2x+1} = 27 \)  
57. \( 5^{2-x} = \frac{1}{125} \)  
58. \( 7^{\frac{x-2}{6}} = \sqrt{7} \)

59. \( 8^{1-x} = 4^{x+2} \)  
60. \( 4^{2x} - 3 \cdot 4^{x} + 2 = 0 \)  
61. \( 3^{2x} + 5 \cdot 3^{x} - 24 = 0 \)
Solve the following logarithmic equations.

62. \( \log_{5} x = 3 \)

63. \( \log_{7} (x + 2) = -2 \)

64. \( \log_{2} \sqrt{x + 4} = 2 \)

65. \( \log_{6} (x + 5) + \log_{6} x = 2 \)

66. \( \log_{2} (x - 1) + \log_{2} (x + 1) = 3 \)

67. \( 2 \log_{2} (x + 4) = \log_{2} 9 + 2 \)

68. \( \log_{10} (x + 7) - \log_{10} 3 = \log_{10} (7x + 1) \)

69. \( \log_{2} (x - 1) - \log_{2} (x + 3) = \log_{2} \left( \frac{1}{x} \right) \)

70. \( \log_{x} 8 + \log_{3} 27 - \log_{3} 81 = -2 \)

Solve the following mixed equations.

71. \( 5^{2x} \cdot 5^{4x} = 125 \)

72. \( \log_{5} 6 - \log_{5} (x + 5) - \log_{5} x = 0 \)

73. \( 3^{x^2 - 12} = 9^{2x} \)

74. \( 8^x = 16^{x+2} \)

75. \( 3^{x+1} = \frac{1}{9^x} \)

76. \( \sqrt{2^{2x+1}} = 4 \)

77. \( x^{\log_{x} 10^8} \)

78. \( \log_{3} \sqrt{x} = \sqrt{\log_{3} x} \)

79. For what values of \( b \) is \( f(x) = \log_{b} x \) an increasing function? a decreasing function?

80. For \( a, b > 0, a, b \neq 1 \), find the value of \( (\log_{a} b) \cdot (\log_{b} a) \).

81. Find values of \( a \) and \( b \) so that the graph of \( f(x) = ae^{bx} \) contains the points \((3,10)\) and \((6,50)\).

82. Suppose that \( x \) is a number such that \( 3^x = 4 \). Find the value of \( 3^{-2x} \).

83. Suppose that \( x \) is a number such that \( 2^x = \frac{1}{3} \). Find the value of \( 2^{-4x} \).

84. Suppose that \( x \) is a number such that \( 2^x = 5 \). Find the value of \( 8^x \).

85. Suppose that \( x \) is a number such that \( 3^x = 5 \). Find the value of \( \left( \frac{1}{9} \right)^x \).

86. For \( b, y > 0 \) and \( b \neq 1, \frac{1}{2} \), show that \( \log_{2b} y = \frac{\log_{b} y}{1 + \log_{b} 2} \).
Find a simplified formula for \( f \circ g \) for the given functions.

87. \( f(x) = \log_6 x, g(x) = 6^{3x} \)

88. \( f(x) = \log_5 x, g(x) = 5^{2+2x} \)

89. \( f(x) = 6^{3x}, g(x) = \log_6 x \)

90. \( f(x) = 5^{3+2x}, g(x) = \log_5 x \)

When solving inequalities, it is common to apply a function to all sides of the inequality. If the function is an increasing function, then the direction of the inequality is preserved, but if the function is a decreasing function, then the direction of the inequality is reversed. Here's why:

If \( f \) is increasing, then for \( x < y \), \( f(x) < f(y) \), and you see that the inequality direction is preserved. If \( f \) is decreasing, then for \( x < y \), \( f(x) > f(y) \), and you see that the inequality direction is reversed.

Here are some examples:

a) To solve the inequality \( 2x < 10 \), you apply the function \( f(x) = \frac{1}{2}x \) to both sides of the inequality. Since it's an increasing function, the direction of the inequality is preserved, and you get \( x < 5 \).

b) To solve the inequality \( -2x < 10 \), you apply the function \( f(x) = -\frac{1}{2}x \) to both sides of the inequality. Since it's a decreasing function, the direction of the inequality is reversed, and you get \( x > -5 \).

c) To solve the inequality \( 1 \leq \frac{1}{3}x < 2 \), you apply the function \( f(x) = 3x \) to all sides of the inequality. Since it's an increasing function, the direction of the inequality is preserved, and you get \( 3 \leq x < 6 \).

d) To solve the inequality \( 1 \leq -\frac{1}{3}x < 2 \), you apply the function \( f(x) = -3x \) to all sides of the inequality. Since it's a decreasing function, the direction of the inequality is reversed, and you get \(-3 \geq x > -6 \implies -6 < x \leq -3 \).

e) To solve the inequality \( 1 \leq 10^x < 5 \), you apply the function \( f(x) = \log_{10} x \) to all sides of the inequality. Since it’s an increasing function, the direction of the inequality is preserved, and you get \( \log_{10} 1 \leq x < \log_{10} 5 \implies 0 \leq x < \log_{10} 5 \).
f) To solve the inequality \(1 \leq \left(\frac{1}{2}\right)^x < 5\), you apply the function \(f(x) = \log_\frac{1}{2} x\) to all sides of the inequality. Since it’s a decreasing function, the direction of the inequality is reversed, and you get \(\log_\frac{1}{2} 1 \geq x > \log_\frac{1}{2} 5 \Rightarrow \log_\frac{1}{2} 5 < x \leq 0\).

Use the previous discussion to solve the following inequalities (Use interval notation.):

91. \(4^{5-x} > 16\)  
92. \(\log_2 \left(\frac{2x-1}{x-2}\right) \leq 0\)  
93. \(5^{x^2-4x} > 5^5\)  
94. \(2^{\frac{3}{x+1}} > 1\)

95. \(8 \leq 4^{5-x} < 16\)  
96. \(-1 < \log_\frac{1}{2} \left(\frac{2x-1}{x-2}\right) \leq 0\)  
97. \((\frac{1}{2})^5 < \left(\frac{1}{3}\right)^{x^2-4x} < 1\)  
98. \(\log_\frac{1}{2} \left(\frac{1}{x-1}\right) > 1\)

Simplify the following as much as possible:

99. \(\log A^3 - \log B^{\frac{2}{3}} + \log A^{\frac{4}{3}} + \log B^{\frac{1}{3}}\)

100. \(\frac{\log(ABC)}{\log(\frac{1}{ABC})}\)

101. \(\frac{\log A^2 - 2\log B}{\log A^2 + \log(\frac{1}{AB})}\)

102. \(10^{\log(AB)}\)

103. \(100^{(\log A-\log B)}\)

104. Find the exact value of the sum \(\log\left(\frac{1}{2}\right) + \log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) + \cdots + \log\left(\frac{999,999}{1,000,000}\right)\).

105. Find the exact value of the sum \(\log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) + \cdots + \log\left(\frac{1,000,000}{999,999}\right)\).

If all the terms of a sequence \(\{x_n\}\) are positive, it is sometimes convenient to analyze the related sequence \(\left\{\frac{x_{n+1}}{x_n}\right\}\). If \(\frac{x_{n+1}}{x_n} \geq 1\), for \(n \geq N\), then the sequence \(\{x_n\}\) is eventually nondecreasing. If \(\frac{x_{n+1}}{x_n} > 1\), for \(n \geq N\), then the sequence \(\{x_n\}\) is eventually increasing. If \(\frac{x_{n+1}}{x_n} \leq 1\), for \(n \geq N\),
then the sequence \( \{x_n\} \) is eventually nonincreasing. If \( \frac{x_{n+1}}{x_n} < 1 \), for \( n \geq N \), then the sequence \( \{x_n\} \) is eventually decreasing. Test the following sequences by analyzing \( \left\{ \frac{x_{n+1}}{x_n} \right\} \).

106. \( x_n = \frac{n}{n+1} \)

107. \( x_n = \frac{n^2 + 1}{n} \)

108. \( x_n = n \left( \frac{3}{4} \right)^n \)

109. \( x_n = \frac{n^2}{2^n} \)

110. \( x_n = \frac{n!}{100^n} \)

111. \( x_n = \frac{1}{n} \left[ \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} \right]^2 \)

112. \( x_n = \frac{2 \cdot 4 \cdots (2n)}{3 \cdot 5 \cdots (2n+1)} \)

113. \( x_n = \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} \)

114. \( x_n = \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} \cdot \frac{1}{2n+2} \)

115. Find an explicit formula for \( a_n \) in the following recursively defined sequence:

\[
a_1 = 1,\ a_{n+1} = a_n - \frac{1}{n(n+1)}.
\]

\(\text{[Hint:} \quad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \text{, so} \quad a_1 = 1 = \frac{1}{2} + \frac{1}{2}, \quad a_2 = 1 - \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{2} + \frac{1}{3}, \quad a_3 = \frac{1}{2} + \frac{1}{3} - \left( \frac{1}{3} - \frac{1}{4} \right) = \frac{1}{2} + \frac{1}{4}, \quad a_4 = \frac{1}{2} + \frac{1}{4} - \left( \frac{1}{4} - \frac{1}{5} \right) = \frac{1}{2} + \frac{1}{5}, \ldots, \text{see if you can formulate the pattern.}\]

116. A sequence \( \{a_n\} \) is given recursively by \( a_{n+1} = 2a_n + n^2 \). If \( a_4 = 23 \), then what’s \( a_1 \)?

117. Find \( x \) so that \( x + 3, 2x + 1, 5x + 2 \) are consecutive terms of an arithmetic sequence.

118. Find \( x \) so that \( 2x, 3x + 2, 5x + 3 \) are consecutive terms of an arithmetic sequence.

119. How many terms must be added in an arithmetic sequence whose first term is 11 and whose common difference is 3 to get the sum 1092?

120. How many terms must be added in an arithmetic sequence whose first term is 78 and whose common difference is -4 to get the sum 702?

121. a) Determine the common difference for the following arithmetic sequence:

\(-5,1,7,13,19,25,\ldots\)
b) Find a formula for \( a_n \) that generates the number in this sequence.

e) Is 71 a number in this arithmetic sequence?

122. a) Determine the common difference for the following arithmetic sequence:

\[ 8, 6, 4, 2, 0, -2, \ldots \]

b) Find a formula for \( a_n \) that generates the numbers in this arithmetic sequence.

c) What is the 99th number in this arithmetic sequence?

123. Find \( x \) so that \( x, x+2, x+3 \) are consecutive terms of a geometric sequence.

124. Find \( x \) so that \( x-1, x, x+2 \) are consecutive terms of a geometric sequence.

125. a) Determine the common ratio for the following geometric sequence:

\[ 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots \]

b) Find a formula for \( a_n \) that generates the numbers in this geometric sequence.

c) Is \( \frac{1}{1024} \) a number in this geometric sequence?

126. a) Determine the common ratio for the following geometric sequence:

\[ 1, -2, 4, -8, 16, -32, \ldots \]

b) Find a formula for \( a_n \) that generates the numbers in this geometric sequence.

c) Is \(-512\) a number in this geometric sequence?

Determine if the following geometric series converge or diverge. If it converges, find its sum.

127. \( 2 + \frac{4}{3} + \frac{8}{9} + \cdots \) 128. \( 6 + 2 + \frac{2}{3} + \cdots \) 129. \( 1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \cdots \)

130. \( 9 + 12 + 16 + \frac{64}{3} + \cdots \) 131. \( \sum_{k=1}^{\infty} 8 \left(\frac{1}{3}\right)^{k-1} \) 132. \( \sum_{k=1}^{\infty} 3 \left(\frac{3}{2}\right)^{k-1} \)

133. \( \sum_{k=1}^{\infty} 4 \left(-\frac{1}{2}\right)^{k-1} \) 134. \( \sum_{k=1}^{\infty} 2 \left(\frac{3}{4}\right)^{k} \)
Prove the following using the Principle of Mathematical Induction:

135. \(1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)\)

136. \(3 + 5 + 7 + \cdots + (2n + 1) = n(n + 2)\)

137. \(1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n - 1)}{2}\)

138. \(1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}\)

139. \(1 + 5 + 5^2 + \cdots + 5^{n-1} = \frac{5^n - 1}{4}\)

140. \(\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n + 1}\)

141. \(1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}\)

142. \(1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \cdots + (2n-1)(2n) = \frac{n(n+1)(4n-1)}{3}\)

143. \(n^3 + 2n\) is divisible by 3

144. \(n(n+1)(n+2)\) is divisible by 6

145. \(1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}\)

146. \(\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots \left(1 + \frac{1}{n}\right) = n + 1\)

147. \(n < 2^n\)

148. \(n^3 - n\) is divisible by 5

149. \(7^n - 4^n\) is divisible by 3.

150. \(\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}\)

151. \(1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + n2^{n-1} = (n-1)2^n + 1\)
152. $5^{2n} - 1$ is divisible by 8.

153. $9^n - 4^n$ is divisible by 5.

154. $2 \cdot 6 \cdot 10 \cdot 14 \cdots (4n - 2) = \frac{(2n)!}{n!}$

155. $n = 3k + 5j$ for some pair of natural numbers, $k$ and $j$, for $n \geq 8$.

156. $(1 - \frac{1}{4})(1 - \frac{1}{9})(1 - \frac{1}{16}) \cdots (1 - \frac{1}{n^2}) = \frac{n + 1}{2n}$; $n \geq 2$.

157. $11^{n+2} + 12^{2n+1}$ is divisible by 133, for $n \geq 0$.

158. $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n} > \frac{13}{24}$; $n \geq 2$.

159. $10^{n+1} - 10(n+1) + n$ is divisible by 81, for $n \geq 0$.

Find a formula for the $n$th term of the following sequences:

160. \( \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots \) \hspace{1cm} 161. \( 2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \ldots \) \hspace{1cm} 162. \( -3, 6, -12, 24, -48, \ldots \)

163. $a_1 = 4, a_{n+1} = .1a_n$ \hspace{1cm} 164. $a_1 = 2, a_{n+1} = -\frac{1}{2}a_n$ \hspace{1cm} 165. $a_1 = 4, a_{n+1} = -a_n$

166. $a_1 = 1, a_{n+1} = \left( \frac{n+1}{2n} \right) a_n$ \hspace{1cm} 167. $a_1 = 1, a_{n+1} = \left( \frac{2a_n + 5}{4} \right)$

168. In a geometric sequence of real number terms, the first term is 3 and the fourth term is 24. Find the common ratio.

169. Find the seventh term of a geometric sequence whose third term is \( \frac{9}{4} \) and whose fifth term is \( \frac{81}{64} \).

170. For what value(s) of $k$ will $k - 4, k - 1, 2k - 2$ form a geometric sequence?

Find the sum of the following series:

171. $\sum_{n=1}^{5} 2^{2-n}$ \hspace{1cm} 172. $\sum_{j=1}^{5} 3(-\frac{1}{3})^{j-1}$ \hspace{1cm} 173. $2 + \frac{2}{3} + \frac{2}{9} + \cdots + \frac{2}{3^{n-1}} + \cdots$

174. $1 + e^{-1} + e^{-2} + e^{-3} + \cdots + e^{-(n-1)} + \cdots$ \hspace{1cm} 175. $\sum_{n=2}^{\infty} \left( \frac{1}{2} \right)^n$ \hspace{1cm} 176. $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$
177. $\sum_{n=0}^{\infty} \left( \frac{5}{2^n} + \frac{1}{3^n} \right)$

178. $\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n}$

179. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$

180. $\sum_{n=0}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^n$

181. $\sum_{n=0}^{\infty} e^{-2n}$

182. $\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n}$

183. Solve the equation $\sum_{i=1}^{n} 2^i = 126$ for $n$.

184. Find the second term of an arithmetic sequence whose first term is 2 and whose first, third, and seventh terms form a geometric sequence.

185. The figure shows the first four of an infinite sequence of squares. The outermost square has an area of 4, and each of the other squares is obtained by joining the midpoints of the sides of the square before it. Find the sum of the areas of all the squares.
186. The equation $x = e^x - 1$ has 0 as its only solution. If the Method of Successive Approximations is applied to approximate this solution, graphically indicate the result if the starting guess is

a) a positive number

b) a negative number
The equation \( x = x^2 - x \) has 0 and 2 as its solutions. If the Method of Successive Approximations is applied to approximate these solutions, graphically indicate the result if the starting guess is

a) greater than 2

b) between 0 and 2
c) less than 0

**188.** Consider the statement \(1 + 2 + 3 + \cdots + n = \frac{(n+2)(n-1)}{2}\).

a) Show that if the statement is true for \(n = k\), then it must be true for \(n = k + 1\).

b) For which natural numbers is the statement true?

**189.** If \(\log_b (3^b) = \frac{b}{2}\), then what’s the value of \(b\)?

**190.** The third term of an arithmetic sequence is 0. Find the sum of the first 5 terms.

**191.** The third term of a geometric sequence is 4. Find the product of the first 5 terms.

**192.** Is there a geometric sequence containing the terms 27, 8, and 12 in any order and not necessarily consecutive? If so, give an example. If not, show why.

**193.** Is there a geometric sequence containing the terms 1, 2, and 5 in any order and not necessarily consecutive? If so, give an example. If not, show why.

**194.** Find all numbers \(a\) and \(b\) so that \(10, a, b, ab\) are the first 4 terms of an arithmetic sequence.
195. Evaluate the sum $11^2 - 1^2 + 12^2 - 2^2 + 13^2 - 3^2 + \cdots + 1000^2 - 990^2$.

\{Hint: \ a^2 - b^2 = (a-b)(a+b).\\}

196. If $4^x + 4^{-x} = 7$, then what is the value of $8^x + 8^{-x}$?

\{Hint: \ 2^x \cdot 4^x + 2^x \cdot 4^{-x} = 7 \cdot 2^x \Rightarrow 8^x + 2^{-x} = 7 \cdot 2^x \text{ and } 2^{-x} \cdot 4^x + 2^{-x} \cdot 4^{-x} = 7 \cdot 2^{-x} \Rightarrow 2^x + 8^{-x} = 7 \cdot 2^{-x}. \ \text{So add the two equations together to get} \ 8^x + 8^{-x} + 2^x + 2^{-x} = 7(2^x + 2^{-x}) \Rightarrow 8^x + 8^{-x} = 6(2^x + 2^{-x}). \ \text{If you knew the value of} \ 2^x + 2^{-x}, \ \text{then you 'd have the answer. Well, what's} \ (2^x + 2^{-x})^2?\}

197. Assuming that $\log_4 6 = A$, $\log_4 10 = B$, and $\log_4 7 = C$, then express the value of $\log_4 112$ using $A$, $B$, or $C$.

\{Hint: \ 112 = 7 \cdot 4^2.\}

198. Assuming that $\log_4 9 = A$, $\log_4 5 = B$, and $\log_4 6 = C$, then express the value of $\log_4 \left( \frac{81}{5} \right)$ using $A$, $B$, or $C$.

\{Hint: \ 81 = 9^2.\}

199. Assuming that $\log_5 7 = A$, $\log_5 12 = B$, and $\log_5 9 = C$, then express the value of $\log_5 245$ using $A$, $B$, or $C$.

\{Hint: \ 245 = 5 \cdot 7^2.\}

200. Assuming that $\log_5 9 = A$, $\log_5 12 = B$, and $\log_5 7 = C$, then express the value of $\log_5 \left( \frac{9}{35} \right)$ using $A$, $B$, or $C$.

\{Hint: \ 35 = 5 \cdot 7.\}

201. Let's prove that $\log_2$ is an irrational number. Suppose that $\log_2 = \frac{m}{n}$ for positive integers $m$ and $n$. Then $10^m = 2$ and therefore $10^m = 2^n$. Compare the ones digits of powers of 10 to the ones digits of powers of 2 to arrive at a contradiction.
202. Let’s prove that $\log_7 5$ is an irrational number. Suppose that $\log_7 5 = \frac{m}{n}$ for positive integers $m$ and $n$. Then $5^m = 7$ and therefore $5^n = 7^m$. Compare the ones digits of powers of 5 to the ones digits of powers of 7 to arrive at a contradiction.

203. Let’s prove that $\log_5 3$ is an irrational number. Suppose that $\log_5 3 = \frac{m}{n}$ for positive integers $m$ and $n$. Then $3^m = 5^n$ and therefore $3^m = 5^n$. Compare the ones digits of powers of 3 to the ones digits of powers of 5 to arrive at a contradiction.

204. a) Show that if $b,d > 0$, then $\frac{a}{b} \leq \frac{c}{d}$ is equivalent to $ad \leq bc$.

{Hint: If $\frac{a}{b} \leq \frac{c}{d}$ then $\frac{a}{b} - \frac{c}{d} \leq 0 \Rightarrow \frac{ad - bc}{bd} \leq 0$, and $ad \leq bc \Rightarrow ad - bc \leq 0 \Rightarrow \frac{ad - bc}{bd} \leq 0$.}

b) Show that if $b,d > 0$ and $\frac{a}{b} \leq \frac{c}{d}$ then $\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$.

c) Use Mathematical Induction to show that if $b_1, b_2, \ldots, b_n > 0$ and $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \ldots \leq \frac{a_n}{b_n}$, then

$$\frac{a_1}{b_1} \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq \frac{a_n}{b_n}.$$  

Use the 1-1 functions $f$ and $g$ which are defined by the following tables to find the following.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>10</td>
<td>-1</td>
</tr>
</tbody>
</table>

205. $f(g(4))$  
206. $(g \circ f)(0)$  
207. $f^{-1}(g(1))$  
208. $f^{-1}(g^{-1}(2))$  
209. $g^{-1}(f^{-1}(-1))$  
210. Solve $g^{-1}(x) = 10$. 

211. Solve \((f \circ g)(x) = 0\).

212. Solve \(f^{-1}(g^{-1}(x)) = 0\).

213. Solve \((f \circ g)(x) > 2\).

214. Solve \(g^{-1}(f(x)) \leq 1\).

215. For the one-to-one functions \(f = \{(0,1),(1,2),(2,3)\} \) and \(g = \{(1,0),(2,1),(4,3)\}\),
\[f \circ g = \{(1,1),(2,2)\} \] and \(g \circ f = \{(0,0),(1,1)\}\), so they could be described as \(f(g(x)) = x\)
and \(g(f(x)) = x\). Are \(f\) and \(g\) inverses? Explain.

216. You might think that if \(f(g(x)) = x\) for all \(x\) in the domain of \(g\) and \(g(f(x))\) is defined
for all \(x\) in the domain of \(f\), that \(g(f(x)) = x\). For \(g = \{(1,2),(3,4)\}\) and
\(f = \{(2,1),(4,3),(6,3)\}, \ f(g(x)) = x\) for all \(x\) in the domain of \(g\), \(g(f(x))\) is defined for
all \(x\) in the domain \(f\). Is it the case that \(g(f(x)) = x\) for all \(x\) in the domain of \(f\)? Explain.

217. A function is called an involution if it is its own inverse. Which of the following functions
are involutions? What type of symmetry does the graph of an involution have?

a) \(f(x) = x\)

b) \(f(x) = -x\)

c) \(f(x) = -x; x \geq 0\)

d) \(f(x) = \frac{1}{x}\)

e) \(f(x) = \frac{1}{x^2}\)

f) \(f(x) = \begin{cases} \sqrt{1-x^2}; & -1 \leq x < 0 \\ -\sqrt{1-x^2}; & 0 < x \leq 1 \end{cases}\)

g) \(f(x) = \begin{cases} x+1; & -1 \leq x < 0 \\ 0; & x = 0 \\ x-1; & 0 < x \leq 1 \end{cases}\)

218. Solve the equation \(\log_{3x} 4 = \log_{2x} 8\).

219. What’s the sum of all the reciprocals of integers whose only prime factors are 2’s, 3’s, and
5’s?
\[
\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 5} + \cdots
\]

{Hint: \((1 + \frac{1}{2})(1 + \frac{1}{3})(1 + \frac{1}{5}) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 5} \cdot \)}

220. How many subsets of the set \(\{2,3,4,5,6,7,8,9\}\) contain at least one prime number?
221. What is the value of \( \sum_{i=1}^{100} \left( \sum_{j=1}^{100} (i + j) \right) \)?

222. What’s the value of \( \frac{111! - 110!}{109!} \)?

223. The first three terms of a geometric sequence are \( \sqrt{3}, 3\sqrt{3}, 9\sqrt{3} \). What’s the fourth term?

224. For what value of \( x \) does \( \log_2 \sqrt{x} + \log_2 x + \log_4 x^2 + \log_8 x^3 + \log_{16} x^4 = 40? \)

225. Suppose that \( a + ar + ar^2 + ar^3 + \cdots \) and \( a + as + as^2 + as^3 + \cdots \) are two different infinite geometric series with the same first term. The sum of the first series is \( r \), and the sum of the second series is \( s \). Find the numerical value of \( r + s \).

226. \( a_1 = 1, \quad a_2 = \frac{1}{\sqrt{3}}, \quad \text{and} \quad a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}}; n \geq 1. \) Find \( a_{2018} \).

227. Three numbers are consecutive numbers of an arithmetic sequence with the first number being 9. If 2 is added to the second number, and 20 is added to the third number, then the three numbers will be consecutive numbers of a geometric sequence. What’s the smallest possible value of the third number of the geometric sequence?

228. Find the domain of the function \( \log_{2018} \left( \log_{2017} \left( \log_{2016} \left( \log_{2015} x \right) \right) \right) \).

229. If \( \log(xy^3) = 1 \) and \( \log(x^2y) = 1 \), then find the value of \( \log(xy) \).
230. In a geometric sequence with positive terms, each term is the sum of the next two terms. What’s the common ratio?

231. Solve the equation \[ \log_4 x - \log_x 16 = \frac{2}{6} - \log_x 8. \]

232. Solve the equation \[ \left( \log_{10} x \right)^{\left( \log_{10} \log_{10} x \right)} = 10,000. \]