Counting Problems for Group 3 (Due by EOC Feb. 20)

A Springtime Path.
1. Determine the number of different paths for spelling the word APRIL:

{Hint: The letters essentially form a tree diagram.}

Being Wholly Positive About The Number Of Divisors.
2. How many distinct positive whole number divisors are there of the integer $30^4 = 810,000$?
{Hint: $30^4 = 2^4 \cdot 3^4 \cdot 5^4$, so every divisor is uniquely determined by}

<table>
<thead>
<tr>
<th>Number of factors of 2</th>
<th>Number of factors of 3</th>
<th>Number of factors of 5</th>
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</table>

Boys Are Icky; No Girls Are Icky.
3. Three boys and 3 girls will sit together in a row.
   a) How many different ways can they sit together without restrictions?

   b) How many different ways can they sit together if the genders must sit together?

   c) How many different ways can they sit together if only the boys must sit together?

   d) How many different ways can they sit together if no two of the same gender can sit together?
4. Three married couples have bought six seats in a row for a performance of a musical comedy.
   a) In how many different ways can they be seated?

   b) In how many different ways can they be seated if each couple must sit together with the husband to the left of his wife?

   c) In how many different ways can they be seated if each couple must sit together?

   d) In how many different ways can they be seated if all the men must sit together and all the women must sit together?

Can You Just Answer My Question?

5. When Professor Sum was asked by Ms. Little how many students were in his class, he answered, “All of my students study either languages, physics, or not at all. One half of them study languages only, one-fourth of them study French, one-seventh of them study physics only, and 20 do not study at all.” How many students does Professor Sum have, if we know he has fewer than 80 students?

   {Hint: The number of students must be a multiple of 2, 4, and 7.}

Consider the sets $A$ and $B$ inside a universal set $U$. $n(U) = n(A \cup B) + n\left[(A \cup B)'ight]$, so we get

$n(U) = n(A) + n(B) - n(A \cap B) + n\left[(A \cup B)'ight]$. This rearranges into

$n(A \cap B) = n(A) + n(B) - n(U) + n\left[(A \cup B)'ight]$, and since $n\left[(A \cup B)'ight] \geq 0$, it must be that $n(A \cap B) \geq n(A) + n(B) - n(U)$. This means that the number of elements in the intersection of $A$ and $B$ is at least $n(A) + n(B) - n(U)$, and it also means that if $n(A) + n(B) - n(U) \leq 0$, then it’s possible that $n(A \cap B) = 0$. This result can be extended to the case of three sets as follows:

$n(A \cap B \cap C) = n\left[A \cap (B \cap C)\right] \geq n(A) + n(B \cap C) - n(U) \geq n(A) + n(B) + n(C) - n(U) - n(U)$

so $n(A \cap B \cap C) \geq n(A) + n(B) + n(C) - 2 \cdot n(U)$. It can further be extended to the case of four sets as follows:

$n(A \cap B \cap C \cap D) = n\left[A \cap (B \cap C \cap D)\right] \geq n(A) + n(B \cap C \cap D) - n(U)$

$\geq n(A) + n(B) + n(C) + n(D) - 2 \cdot n(U) - n(U)$

so

$n(A \cap B \cap C \cap D) \geq n(A) + n(B) + n(C) + n(D) - 3 \cdot n(U)$.
In general, you can show that
\[
n(A_1 \cap A_2 \cap \cdots \cap A_k) \geq n(A_1) + n(A_2) + \cdots + n(A_k) - (k - 1) \cdot n(U).
\]
Also, \(n(A \cap B) \leq n(A)\) and \(n(A \cap B) \leq n(B)\), so \(n(A \cap B) \leq \min[n(A), n(B)]\). In general, you can show that
\[
n(A_1 \cap A_2 \cap \cdots \cap A_k) \leq \min[n(A_1), n(A_2), \ldots, n(A_k)].
\]

If You Can’t Work On Transmissions, That’s The Brakes.

6. A car shop has 12 mechanics, of whom 8 can work on transmissions and 7 can work on brakes.
   a) What is the minimum number who can do both?

   b) What is the maximum number who can do both?

   c) What is the minimum number who can do neither?

   d) What is the maximum number who can do neither?

   *Hint: See the previous discussion.*

War Is Hell!

7. In a group of 100 war veterans, if 70 have lost an eye, 75 an ear, 80 an arm, and 85 a leg:
   a) at least how many have lost all four?

   *{Hint: See the hint for problem 6.}*

   b) at most how many have lost all four?

Even You Can Choose Two.

8. In how many different ways can you select two distinct integers from the set \(\{1, 2, 3, \ldots, 100\}\) so that their sum is even?

   *{Hint: What kinds of numbers will produce even sums? even + even = ?, odd + odd = ?, even + odd = ?}*
The ABC’s Of The Universe.

9. $A$, $B$, and $C$ are three sets with $U = A \cup B \cup C$. Use the given Venn diagram to answer the following:

\[
\begin{array}{c}
\text{A} \\
29 \\
9-x \\
7-x \\
C \\
13+x \\
\text{B} \\
23 \\
x \\
16 \\
\text{C} \\
\end{array}
\]

\[
U
\]

a) Find $n(B)$.

b) Given that $n(A) = 41$, find $x$.

c) Find $n(A' \cap B \cap C)$.

Shake It Like You Mean It.

10. a) Twenty people are at a party. If everyone at the party shakes the hand of everyone else at the party, determine the total number of handshakes.

b) Ten married couples are having a party. If each person at the party shakes the hand of everyone else except his/her spouse, determine the number of handshakes at the party.

It’s A Monthly Thing.

11. a) What is the smallest number of people in a group that will guarantee that at least two of the people were born in the same month?

b) What is the smallest number of people in a group that will guarantee that at least three of the people were born in the same month?
Texas Hold’em.

12. In this problem, we’ll determine the number of possible particular 5-card poker hands. Here is a possible decision process for the 5-card poker hand with one pair:

<table>
<thead>
<tr>
<th>13</th>
<th>4 ( \binom{2}{4} )</th>
<th>12 ( \binom{3}{4} )</th>
<th>4</th>
<th>4</th>
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<tr>
<td>Which kind of pair?</td>
<td>Which two cards of this kind?</td>
<td>Which 3 other kinds?</td>
<td>Which one of the first kind?</td>
<td>Which one of the second kind?</td>
<td>Which one of the third kind?</td>
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So there are \( 13 \cdot 4 \binom{2}{4} \cdot 12 \binom{3}{4} \cdot 4 \cdot 4 \cdot 4 = 1,098,240 \) different two-of-a-kind 5-card poker hands.

a) See if you can do the same thing to find the number of different three-of-a-kind hands:

| Which kind of three-of-a-kind? | Which three cards of this kind? | Which 2 other kinds? | Which one of the first kind? | Which one of the second kind? |

b) See if you can do the same thing to find the number of different four-of-a-kind hands:

| Which kind of four-of-a-kind? | Which other kinds? | Which one of the other kind? |

The number of different flushes, i.e. five cards of the same suit, but not in order
First we’ll count the number of different hands with 5 cards of the same suit:

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<th>4</th>
<th>13 ( \binom{5}{5} )</th>
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<td>Which suit?</td>
<td>Which 5 cards?</td>
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Then we’ll subtract the number of 5-card hands in order of the same suit (straight flushes):

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<tr>
<td>Which suit?</td>
<td>Which kind of card starts the straight flush?</td>
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So we get \( 4 \cdot 13 \binom{5}{5} - 4 \cdot 10 = 5,108 \) different 5-card poker hands which are flushes.

b) See if you can do something similar to find the number of straights, i.e. 5 cards in a row, but not all of the same suit.
First we’ll count the number of different hands with 5 cards in a row:

| Which kind of card starts the straight? | Which suit for the first card? | Which suit for the second card? | Which suit for the third card? | Which suit for the fourth card? | Which suit for the fifth card? |

Then we’ll subtract the number of 5-card hands in order of the same suit (straight flushes):
You have learned that the number of permutations of $n$ distinct objects is $n!$. For instance if you wanted to seat three people along one side of a rectangular table, the number of possible arrangements is $3!$. However, if the three people are to be seated around a circular table, the number of possible arrangements is only $2!$. Let’s see why: If the people are labeled A, B, and C, the two arrangements look like the following:

At first, it might seem that there should be $3! = 6$ different arrangements, like the following:

But, if you look closely, you’ll see that arrangements (1), (4), and (5) are identical, each is just a rotation of the other. The same is true of (2), (3), and (6).
Knights Of The Circular Table And The Venerable Bead.

13. a) Find a formula for the number of different ways that n people(or objects) can be seated(or placed) around a circular table.

   \(\text{Hint: Start with } n!, \text{ but divide it by the number of rotations that can be made that generate equivalent arrangements.}\)

b) Use the previous formula to find the number of different arrangements of 12 people around a circular table.

c) Use the previous formula to find the number of different necklaces that use 10 different colored beads.

d) Modify the previous formula to find the number of different necklaces that use 20 beads with 5 red, 4 blue, 8 green, and 3 yellow.

   \(\text{Hint: Modify an idea from permutations of non-distinguishable objects.}\)

Just Be A Sport.

14. Thirty-one students participate in baseball, soccer, and tennis. Some play only one sport, some play two sports, and a few play all three. The results are:
   19 play baseball
   16 play soccer
   17 play tennis
   9 play baseball and soccer
   10 play soccer and tennis
   8 play baseball and tennis
   6 play all three sports

a) How many students participate in only one sport?

b) How many students participate in exactly two sports?
15. a) Find the number of ways that seven red marbles and eight white marbles can be placed into 3 boxes if each box contains at least one of each color.

b) Find the number of ways that seven red marbles and eight white marbles can be placed into 3 boxes if some of the boxes might not have each color or may be empty.

{Hint: If we assume that each box receives at least one red then we’ll decide how many each box gets by choosing 2 spaces from the 6 spaces between the 7 red marbles:

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Box # 1 gets 1 red marble

Box # 2 gets 3 red marbles

Box # 3 gets 3 red marbles

So if each must box gets at least one, there are \( \binom{7}{2} = 21 \) different ways that the 7 red marbles could have been distributed into the 3 boxes. To allow for the possibility that one or more boxes don’t get any red marbles, we’ll pretend that there are actually 10 red marbles for the 3 boxes.

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Box # 1 gets 2 red marbles

Box # 2 gets 2 red marbles

Box # 3 gets 7 red marbles

From the 9 spaces available, we’ll select 2. If we subtract 1 from each number of red marbles assigned to each box, we’ll have a way that the boxes could contain all 7 red marbles even if some don’t have any.

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Box # 1 gets 1 red marble

Box # 2 gets no red marbles

Box # 3 gets 6 red marbles
It’s As Easy As ABCDEFG.

16. Find the number of permutations of ABCDEFG that contain the following:
   a) the sequence ABC

   \{Hint: They would look like one of the following:\}

   \[
   \begin{array}{cccc}
   A & B & C & \text{ } \\
   A & B & C & \text{ } \\
   \text{ } & A & B & C \\
   \text{ } & \text{ } & A & B & C \\
   \text{ } & \text{ } & \text{ } & A & B & C
   \end{array}
   \]

   b) the sequences AB, CD, and EF, but not necessarily in this order.

   \{Hint: Treat each pair of letters as a single unit, and decide the position of G. For example,\}

   \[
   \begin{array}{cccc}
   AB & CD & EF & G \\
   AB & CD & G & EF \\
   AB & G & CD & EF \\
   G & AB & CD & EF
   \end{array}
   \]

   c) the sequences AB, BC, and EF, but not necessarily in this order.

Have A Seat!

17. In a classroom, there are 28 chairs. If 26 students are to be seated in the classroom, how many different ways can this be done?

   \{Hint: Rather than assign students to seats, assign seats to students.\}
**Do College Students Still Read?**

18. A survey of 100 college students revealed the following results:

- 40 read Time Magazine
- 30 read Newsweek Magazine
- 25 read U.S. News and World Report Magazine
- 15 read Time and Newsweek Magazines
- 12 read Time and U.S. News and World Report Magazines
- 10 read Newsweek and U.S. News and World Report Magazines
- 4 read all three magazines

a) How many read at least one magazine?  
b) How many read exactly one magazine?  
c) How many read exactly two magazines?  
d) How many read none of these magazines?

{Hint: Make a Venn diagram.}

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**Well Jenny, It Can’t Be 867-5309!**

19. John is having trouble remembering his girlfriend Jenny’s 7-digit phone number. He remembers that the first four digits consist of one 1, one 2, and two 3s. He also remembers that the fifth digit is either a 4 or 5. While he has no memory of the sixth digit, he remembers that the seventh digit is 9 minus the sixth digit. If this is all the information he has, how many possible phone numbers are there?

{Hint: }

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**Don’t Be A Sloth-Two Or Three Toed.**

20. a) How many whole numbers less than 1,000 contain no digits of 3 but at least one digit of 2?

{Hint: How many have no 3’s? How many have no 2’s?}

b) How many whole numbers less than 1,000,000 contain no digits of 3 but at least one digit of 2?