4.5: Inverse of a Square Matrix

Identity matrix for multiplication:

For real numbers, 1 is the identity for multiplication.

Is there an identity for matrix multiplication? i.e. is there a matrix $I$ such that $MI = IM = M$?

However, for square matrices, there is such an identity.

For $n \times n$ matrices, $I$ is the matrix with 1 on the principal diagonal and zeros elsewhere.

Examples of Identity Matrices:

Inverses:

Every real number except 0 has a multiplicative inverse.

The Inverse of a Square Matrix:

Let $M$ be an $n \times n$ square matrix and $I$ be the $n \times n$ identity matrix. If there exists a matrix $M^{-1}$ such that $M^{-1}M = MM^{-1} = I$, then $M^{-1}$ is the inverse of $M$. 
Example 1: Verify that \[
\begin{bmatrix}
5 & 3 \\
1 & 1 \\
\end{bmatrix}
\] and \[
\begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{5}{2} \\
\end{bmatrix}
\] are inverses of one another.

How to find the inverse:

- To find the inverse of a matrix \( M \), start by creating an augmented matrix \( [M \mid I] \) by placing the appropriate-sized identity matrix to the right of the vertical line.
- Then row-reduce the augmented matrix until the identity matrix appears to the left of the vertical line. Then \( M^{-1} \) is to the right of the vertical line. In other words, row-reduce your augmented matrix until it looks like \( [I \mid M^{-1}] \).
- If a zero row appears to the left of the vertical line, then \( M^{-1} \) does not exist.

Example 2: Find the inverse of \( M = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \), if it exists.
Example 3: Find the inverse of $M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, if it exists.

Example 4: Find the inverse of $M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, if it exists.
Example 5: Find the inverse of \( M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \), if it exists.

Shortcut for \( 2 \times 2 \) matrices:

Let \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). Then the determinant of \( M \) is \( D = ad - bc \). If \( D \neq 0 \), then \( M^{-1} \) exists and is given by

\[
M^{-1} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

Example 6: Find the inverse of \( M = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix} \), if it exists.
**Example 7:** Find the inverse of \( M = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \), if it exists.

**Example 8:** Find the inverse of \( M = \begin{bmatrix} 3 & 5 \\ 2 & 8 \end{bmatrix} \), if it exists.