11.5: Normal Distributions

Up to now, we’ve dealt with discrete random variables, variables that take on only a finite (or countably infinite—we didn’t do these) number of values.

A continuous random variable takes on all real values in an interval.

Examples: Continuous random variables:
Heights of people, lifetimes of light bulbs, lengths of phone calls.

Discrete random variables:
Number of defective clocks in a sample, number of students who say they like math, houses in a neighborhood.

For many continuous random variables, most of the data points fall near the center, while fewer data points fall near the ends. Such probability distributions resemble a normal distribution. The graph of a normal distribution is sometimes called a normal curve, or sometimes a bell curve.

Examples of normal curves:
The normal distribution is the most important distribution in mathematics, science, and many other fields. It serves as a very good approximation for many other distributions, including the binomial distribution for large $n$.

**Areas under the normal curve:**

One of the many remarkable things about normal curves is the use of area under a normal curve. No matter what the shape of a normal curve, specific areas (and thus probabilities) can be determined easily by using only one table.

**The standard normal curve:**

The standard normal curve is the normal curve with mean 0 and standard deviation 1. Other normal curves are related to the standard normal curve in this way: the area under any normal curve from $\mu$ to $\mu + z\sigma$ corresponds to the area under the standard normal curve from 0 to $z$.

So we convert our values to $z$-values, usually called $z$-scores. $Z$-scores tell us how many standard deviations a data value is away from the mean.

**$z$-scores:**

In a normal distribution with mean $\mu$ and standard deviation $\sigma$, where $x$ is a data value, the $z$-score is

$$z = \frac{x - \mu}{\sigma}.$$  

The area under a normal curve between $x = a$ and $x = b$ is the same as the area under the standard normal curve between the $z$-score for $a$ and the $z$-score for $b$. 

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**Normal Curve Properties:**

1. A normal curve is bell-shaped and symmetric about a vertical line.
2. The mean is on the horizontal axis at the axis of symmetry.
3. The shape is completely determined by the mean $\mu$ and the standard deviation $\sigma$.
   - Small $\sigma$: tall, narrow curve
   - Large $\sigma$: flat, broad curve
4. The area between the curve and the horizontal axis is always 1. (This corresponds to the fact that all the probabilities in a distribution must add up to 1.)
5. Regardless of the shape,
   - 68.3% of the area is within one standard deviation from the mean.
   - 95.4% of the area is always within two standard deviations from the mean
   - 99.7% of the area is always within three standard deviations from the mean.
**Example 1:** Consider a normal distribution with mean 10 and standard deviation 3.

a. What is the $z$-score corresponding to $x = 28$?

b. What is the $z$-score corresponding to $x = 7$?

**Important:** The $z$-score is the number of standard deviations between the data point and the mean.

To determine areas under the curve and thus probabilities, we’ll use a table. (See Table 1 in Appendix C, p. 656) Probabilities of $x$ falling inside a certain range of values are given by the area under the normal curve for that range.

**Example 2:** Consider a normal curve with mean 7 and standard deviation 2.

a. What is the area under the curve between 7 and 10?

b. What is the probability that the variable is between 7 and 10?

c. What is the probability that the variable is between 6.5 and 9.7?

d. What is the probability that the variable is less than 4.52?
Example 3: Dusty Dog Food Company ships dog food to its distributors in bags whose weights are normally distributed with a mean weight of 50 pounds and standard deviation 0.5 pound. If a bag of dog food is selected at random from a shipment, what is the probability that it weighs

a. More than 51 pounds?

b. Less than 49 pounds?

c. Between 49 and 51 pounds?

Properties of Normal Probability Distributions:

1. \( P(a \leq x \leq b) = \text{area under the curve from } a \text{ to } b. \)
2. \( P(-\infty \leq x \leq \infty) = 1 = \text{total area under the curve.} \)
3. \( P(x = c) = 0. \)

Note: \( P(a \leq x \leq b) = P(a \leq x < b) = P(a < x \leq b) = P(a < x < b) \)
Example 4: The medical records of infants delivered at a certain hospital show that the infants’ birth weights in pounds are normally distributed with a mean of 7.4 and a standard deviation of 1.2.

a) Find the probability that an infant selected at random from among those delivered at the hospital weighed more than 9.2 pounds at birth.

b) Find the probability that an infant selected at random from among those delivered at the hospital weighed less than 8 pounds at birth.

c) Find the probability that an infant selected at random from among those delivered at the hospital weighed between 8 and 10 pounds.
Example 5:  The GPA of the senior class of a certain high school is normally distributed with a mean of 2.7 and a standard deviation of 0.4 point. If a senior in the top 10% of his or her class is eligible for admission to any state university, what is the minimum GPA that a senior should have to ensure eligibility to a state university?

Using a normal distribution to approximate a binomial distribution:

When working with a binomial distribution, the calculations can be long and tedious, especially with a large number of trials \((n)\).

For a large number of trials, a binomial histogram resembles a normal curve. In many cases, we can use a normal distribution to closely approximate a binomial distribution.

In order to approximate the binomial distribution by a normal distribution, we need to know its mean and standard deviation.

<table>
<thead>
<tr>
<th>Mean and Standard Deviation of the Binomial Distribution:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean: ( \mu = np )</td>
</tr>
<tr>
<td>Standard Deviation: ( \sigma = \sqrt{npq} )</td>
</tr>
</tbody>
</table>

\( n \) = number of trials  
\( p \) = probability of success on one trial  
\( q = 1 - p \) = probability of failure on one trial

Note: The mean is the expected number of successes in \( n \) trials. So we could write \( E(x) = np \), where \( x \) is the number of successes in \( n \) trials.
The “0.5 adjustment”:

When using a normal distribution to approximate a binomial distribution, we need to adjust for the fact that our binomial distribution is represented by a histogram with frequency classes.

For example, the data value “7” represents 7 successes. The corresponding bar on the histogram will cover the values 6.5 to 7.5.

Example 6: Suppose that a basketball player has a 73% free throw shooting percentage. What is the probability that he is successful in at least 100 of his next 120 attempts?
Example 7: The probability that a heart transplant performed at a certain hospital is successful (that is, the patient survives 1 year or more after undergoing the surgery) is 0.7. 100 patients have undergone such an operation. Use a normal distribution to approximate the probability that

a. Fewer than 75 will survive 1 year or more after the operation.

b. Between 80 and 90, inclusive, will survive 1 year or more.